Complex analytic spaces

lecture 18: Chow theorem and other applications of Remmert-Stein

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Remmert and Remmert-Stein theorem (reminder)

DEFINITION: A proper map is a continuous map such that a preimage of a compact is always compact.

THEOREM: (Remmert-Stein theorem)
Let X be a complex variety, $A \subset X$ a complex analytic subset, and Z an irreducible complex analytic subset in $X \backslash A$. Assume that $\dim Z > \dim A$. **Then the closure of Z is complex analytic in X.**

EXERCISE: Deduce the Riemann removable singularity theorem from Remmert-Stein, in the following form. **Let f be a continuous function on $\mathbb{C}$, holomorphic outside of a discrete set. Prove that f is holomorphic.**

THEOREM: (Remmert proper map theorem)
Let $F: X \longrightarrow Y$ be a proper morphism of complex varieties. **Then $F(X)$ is complex analytic in Y.**

Chow theorem

DEFINITION: The set of common zeroes of an ideal in the graded ring of homogeneous polynomials on $\mathbb{C}^{n+1}$ is called **a projective variety**.

THEOREM: (Chow)
Let $Z \subset \mathbb{C}P^n$ be a closed complex-analytic subset. Then Z is projective.

Proof. Step 1: Consider the natural projection $\mathbb{C}^{n+1}\backslash 0 \longrightarrow \mathbb{C}P^n$, let $C_0(Z) \subset \mathbb{C}^{n+1}\backslash 0$ be the preimage of Z, and $C(Z) \subset \mathbb{C}^{n+1}$ its closure. Remmert-Stein (applied to $A = \{0\}$, $X = \mathbb{C}^{n+1}$, $Z = C_0(Z)$) **imply that $C(Z)$ is complex analytic.**
Step 2: Let $I \subset \mathcal{O}_{n+1}$ be the ideal of $C(Z)$. The group $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}$ by homotheties. Since $C(Z)$ is $\mathbb{C}^*$-invariant, **the ideal I_Z is $\mathbb{C}^*$-invariant.**

Step 3: Let $f \in I$, and $f = \sum P_i$ its Taylor series decomposition, with $P_i \in \mathbb{C}[z_1, ..., z_{n+1}]$ being homogeneous polynomials of degree i. Then the action ρ_λ of $\mathbb{C}^*$ takes f to $\rho_\lambda^*(f) = \sum \lambda^i P_i$. For any point $z \in C(Z)$, consider the function $\rho_\lambda^*(f)(z) = \sum \lambda^i P_i(z)$. Since $f \in I$ and $C(Z)$ is $\mathbb{C}^*$-invariant, we have $\rho_\lambda^*(f)(z) = 0$, hence all coefficients of the Taylor series for the function $\lambda \mapsto \rho_\lambda^*(f)(z)$ vanish. **This implies that $P_i(z) = 0$ for all $z \in C(Z)$,** hence by Rückert Nullstellensatz $P_i \in I$.
Step 4: Let I_{gr} be the ideal generated by all homogeneous polynomials vanishing in $C(Z)$. Using Step 3, **we obtain that $z \in C(Z)$ if and only if z belongs to the zero set of I_{gr}.** Therefore, Z is projective. ■

Chow Wei-Liang

Chow Wei-Liang: October 1, 1911, Shanghai - August 10, 1995, Baltimore

Family portrait in Shanghai, in an undated photo (prior 1949), standing: Zhou Wei Liang and his wife Margot Victor. Seating from L: Weiliang's Mom Wanjun Yu, his daughters Marian (1937 Shanghai -) and Margaret (1940 Shanghai -), and his Dad MD Chow.

Chow Wei-Liang was also a stamp collector, known for his book "Shanghai Large Dragons", The First Issue of The Shanghai Local Post, published in 1996.

Smooth points of irreducible varieties

THEOREM: Let X be an irreducible complex variety, and $X_{ns} \subset X$ the set of smooth points. **Then X_{ns} is connected.**

Proof. Step 1: Let X_{sing} be the set of singular points of X. Then $\dim X_{\text{sing}} < \dim X$. Let X_1 be the connected components of X_{ns} which has maximal dimension, $\dim X_1 = \dim X$, and $\overline{X}_1$ its closure in X. Since $\overline{X}_1 \backslash X_1 \subset X_{\text{sing}}$, this set is contained in a complex variety of dimension $< \dim X_1$. **This implies that X_1 is complex analytic in a complement to a variety of dimension $< \dim X_1$.**. By Remmert-Stein, $\overline{X}_1$ is complex analytic in X.

Step 2: Since X is irreducible, $\overline{X}_1$ is a proper subvariety of X. This is impossible, because $\dim X = \dim \overline{X}_1$. ■

COROLLARY: Any open subset of an irreducible complex variety is equidimensional.

Complex analytic maps are algebraic

REMARK: Recall that **a regular function** on a projective variety $X \subset \mathbb{C}P^n$ is a (locally defined) function which is polynomial in all affine charts. Clearly, regular functions form a sheaf. An **algebraic morphism** of complex projective varieties is a map $\varphi: X \longrightarrow Y$ such that a pullback of a regular function is always regular. In other words, the algebraic morphism is a morphism given by polynomial maps in affine coordinates.

THEOREM: Let $f: X \longrightarrow Y$ be a complex analytic map of complex manifolds. **Then f is algebraic.**

Proof: Let $\Gamma_f \subset X \times Y$ be the graph of f. Using the Segre embedding $\mathbb{C}P^n \times \mathbb{C}P^m \hookrightarrow \mathbb{C}P^{(n+1)(m+1)-1}$, we can consider $X \times Y$ as an subvariety in $\mathbb{C}P^{(n+1)(m+1)-1}$. Chow theorem implies that Γ_f is projective, and the projection maps $\Gamma_f \longrightarrow X$ and $\Gamma_f \longrightarrow Y$ are regular. A bijective regular map of complex projective manifolds is an isomorphism in the category of algebraic varieties by Zariski main theorem. Therefore, the projection $X \xrightarrow{\sim} \Gamma_f \longrightarrow Y$ (which is equal to f) is an algebraic morphism. ■

Algebraic structures

DEFINITION: **An algebraic structure** on a complex variety X is a subsheaf $\mathcal{O}_X^{reg} \subset \mathcal{O}_X$ of a sheaf of holomorphic functions, such that there exists an algebraic variety $\mathcal{X}$ such that X is its maximal spectrum, and $\mathcal{O}_X^{reg}$ its sheaf of regular functions. It is called **projective algebraic** if $\mathcal{X}$ is projective.

COROLLARY: **Any two projective algebraic structures on a compact complex manifold X are equal.**

Proof: Indeed, for any two projective algebraic structures X_1, X_2, the natural bijection between these spaces induces an equivalence between the sheaves of regular functions, by the previous theorem. ■

REMARK: This is not so when X is not compact. Indeed, by a theorem of Z. Jelonek (arXiv:1307.5564), for any affine non-rational curve V, **the manifold $\mathbb{C}\times V$ has uncountably many pairwise non-equivalent algebraic structures.**

REMARK: In dimension 1, all compact complex varieties admit a projective algebraic structure. In dimension 2 or more, **there are compact complex manifolds not admitting algebraic structures.**

Irreducible varieties are connected

PROPOSITION: Let X be an irreducible projective variety, and $Y \subsetneq X$ a complex subvariety. **Then the complement $X \backslash Y$ is connected.**

Proof: Let X_1, X_2, ..., X_n be connected components of X, Then X_i is complex analytic in $X \backslash Y$, where $\dim Y < \dim X_i$, hence by Remmert-Stein theorem its closure $\overline{X}_i$ is complex analytic. **Then $X = \bigcup_i \overline{X}_i$, which is impossible, because X is irreducible.** ■

Meromorphic functions and their graphs

PROPOSITION: Let f be a meromorphic function on a complex variety X, and Γ_f be its graph, considered as a subvariety in $(X\backslash P)\times\mathbb{C}$, where P is the pole set of f. **Then the closure of Γ_f in $X\times\mathbb{C}P^1$ is complex analytic.** Conversely, **any complex analytic subset $\Gamma\subset X\times\mathbb{C}P^1$ which is projected to X bimeromorphically is obtained this way.**

Proof. Step 1: Let D be the zero set of f; this is the same as the pole set of f^{-1}. The graph of f^{-1} can be considered as a subset of $(X\backslash D)\times\mathbb{C}$. If we pass from one affine chart on $X\times\mathbb{C}P^1$ to another chart, the graph of Γ_f becomes the graph of $\Gamma_{f^{-1}}$. Gluing these two graphs, we obtain a complex analytic subset $\tilde\Gamma\subset(X\backslash D\cap P)\times\mathbb{C}P^1$. Let $A\subset X\times\mathbb{C}P^1$ be $D\cap P\times\mathbb{C}P^1$ **Since $\dim D\cap P=\dim X-2$, we can apply Remmert-Stein to $Z=\tilde\Gamma$ and $A=D\cap P\times\mathbb{C}P^1$** and obtain that the closure of $\tilde\Gamma$ is complex analytic.

Step 2: Conversely, any $\Gamma\subset X\times\mathbb{C}P^1$ which is projected to X bimeromorphically defines a meromorphic map to $\mathbb{C}P^1$, hence a meromorphi map to $\mathbb{C}$, which is the same as a meromorphic function. ■

Meromorphic functions are rational

REMARK: Let X be a projective manifold. Recall that a meromorphic function f on X is called **rational** if it is a quotient of two regular functions.

THEOREM: Let f be a meromorphic function on a projective variety X. **Then it is rational.**

Proof: The graph $\Gamma_f \subset X \times \mathbb{C}P^1$ is complex analytic, hence projective; the corresponding coordinate functions are regular, and their quotient is rational.
■