Complex analytic spaces

lecture 19: Regular local rings and Auslander-Buchsbaum theorem

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Spectrum of a ring (reminder)

DEFINITION: **Spectrum** of a ring R is the set $\operatorname{Spec} R$ if its prime ideals.

DEFINITION: Let $J \subset R$ be an ideal, and $V(J) \subset \operatorname{Spec} R$ be the set of all prime ideals containing J. **Zariski topology** on $\operatorname{Spec} R$ is topology where all $V(J)$ (and only those) are closed. Clearly, $V(J_1) \cap V(J_2) = V(J_1 + J_2)$ and $V(J_1) \cup V(J_2) = V(J_1 J_2)$, hence finite unions and intersections of closed sets are closed.

DEFINITION: Let R be a ring, $\operatorname{Spec}(R)$ its spectum and $f \in R$. **Affine open set** is an open set $U_f := \operatorname{Spec} R \backslash V_{(f)}$. We identify U_f with $\operatorname{Spec}(R[f^{-1}])$ (localization in f).

EXERCISE: Prove that **finite intersection of affine open sets is affine, $U_f \cap U_g = U_{fg}$.**

EXERCISE: Prove that **affine open sets give a base of Zariski topology.**

Affine schemes (reminder)

DEFINITION: The sheaf $\mathcal{O}$ of regular functions on $\operatorname{Spec} R$ is defined as the sheaf which satisfies $\mathcal{O}\big|_{U_f} = R[f^{-1}]$, with restriction maps taking a function to its restriction to an open set.

EXERCISE: Prove that $\mathcal{O}\big|_{U_f} = R[f^{-1}]$ **is sufficient to define a sheaf,** which is reconstructed uniquely from this property.

DEFINITION: A scheme is a ringed space $(M, \mathcal{O})$, which is locally isomorphic to an affine scheme with the sheaf of regular functions. In this situation sheaf $\mathcal{O}$ is called **the structure sheaf** of the scheme, or **the sheaf of regular functions**.

REMARK: The structure sheaf **may contain nilpotents. An algebraic variety** is a scheme which does not have nilpotents in its structure sheaf.

DEFINITION: Morphism of affine schemes is a morphism of ringed spaces $\operatorname{Spec} A \longrightarrow \operatorname{Spec} B$ induced by a ring homomorphism $B \longrightarrow A$. **Morphism of schemes** is a map of schemes which is given by morphisms of affine schemes in local affine charts.

Zariski main theorem

We have used the following theorem.

THEOREM: Let $f: X \longrightarrow Y$ be a bijective morphism of complex projective manifolds. **Then f is an isomorphism of algebraic varieties.**

Its proof is deduced from two results.

THEOREM: $f: X \longrightarrow Y$ be a dominant morphism of irreducible algebraic varieties. Assume that $f^{-1}(Y)$ is one point for a Zariski dense subset of Y. **Then f induces an isomorphism of fraction fields $k(Y) \longrightarrow k(X)$.**

THEOREM: (Zariski main theorem)
Let $f: X \longrightarrow Y$ be a regular, birational morphism of algebraic manifolds. **Then either f is an open embedding, or there exists a divisor $E \subset X$ such that its image has dimension $\leqslant \dim X - 2$.**

Zariski main theorem is deduced from another fundamental theorem

THEOREM: (Auslander-Buchsbaum theorem)
Let $x \in M$ be a smooth point on an algebraic variety, and $\mathcal{O}_{M,x}$ its local ring. **Then $\mathcal{O}_{M,x}$ is factorial**.

Today we prove Auslander-Buchsbaum.

Maurice Auslander (1926-1994)

In 1958, Masayoshi Nagata has proven that all regular local rings are factorial if all 3-dimensional regular local rings are factorial. In 1959, Maurice Auslander and David Buchsbaum proved that all 3-dimensional regular local rings are factorial.

Masayoshi Nagata (1927-2008)

David Buchsbaum and Maurice Auslander

David Buchsbaum (1929-2021)

David and Betty Buchsbaum, New York, 1950.

Maurice Auslander (1926-1994)

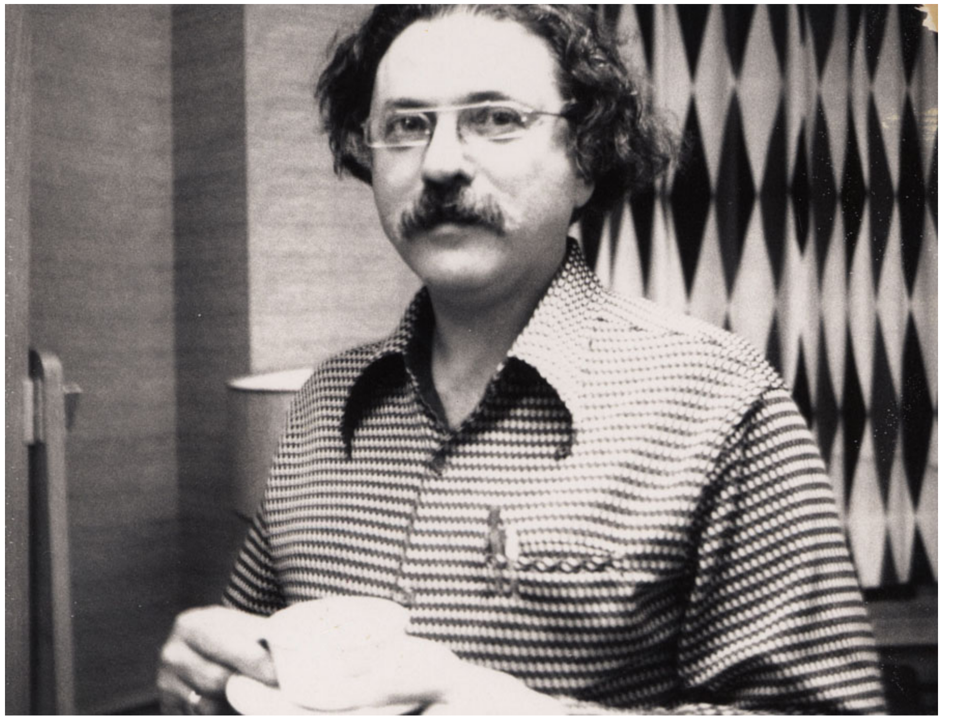

photo by Paul Halmos, Feb. 18, 1974 Indiana University in Bloomington

Smooth points of complex varieties

PROPOSITION: Let $x \in X$ be a point in an n-dimensional complex analytic variety, and $\mathfrak{m}_x$ its maximal ideal. **Then** $\dim_{\mathbb{C}} \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2} \geqslant n$**.** Moreover, x **is a smooth point if and only if** $\dim_{\mathbb{C}} \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2} = n$**.**

Proof. Step 1: Assume that $\dim_{\mathbb{C}} \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2} = n$. We need to show that X is smooth. Since the statement is local, we may assume that $X \subset \mathbb{C}^m$ is a closed analytic subvariety. Let $J \subset \mathcal{O}_m$ be the ideal of X, and $h_1, ..., h_s$ its generators. A point $z \in X$ is smooth if the space generated by $dh_1|_z, ..., dh_s|_z \in T_z^*\mathbb{C}^n$ is $m-n$-dimensional. Let $f_1, ..., f_n \in \mathfrak{m}$ generate $\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$. Then the ideal of $z \in \mathbb{C}^m$ is generated by $h_1, ..., h_s$ and $f_1, ..., f_n$; **therefore, the space generated by** $dh_1|_z, ..., dh_s|_z$ **and** $f_1, ..., f_n$ **is** m**-dimensional.** This gives $\dim\langle dh_1|_z, ..., dh_s|_z\rangle \geqslant m-n$. Then $\dim\langle dh_1|_z, ..., dh_s|_z\rangle = m-n$ if and only if x is smooth (Lecture 12, page 14, step 2)

Step 2: If $\dim_{\mathbb{C}} \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2} < n$, by the same argument we obtain that $\dim\langle dh_1|_z, ..., dh_s|_z\rangle = t > m-n$. Assume that the first t differentials are linearly independent: $\dim\langle dh_1|_z, ..., dh_t|_z\rangle = t$. **Then the map** $H(z) := (h_1(z), ..., h_t(z))$ **is a holomorphic submersion, and** X **is a subvariety in** $m-t$**-dimensional manifold.** This is impossible because $t > m-n$ and $m-t < n$. ■

Regular local rings

DEFINITION: Recall that **the Krull dimension** $\dim R$ of a ring R is n, where $n+1$ the length of the largest chain of prime ideals

$$\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq ... \subsetneq \mathfrak{p}_{n+1} \subsetneq R.$$

DEFINITION: A regular local ring is a Noetherian local ring R with maximal ideal $\mathfrak{m}$ and the residue field $k := R/\mathfrak{m}$ such that $\dim_k \frac{\mathfrak{m}}{\mathfrak{m}^2} = \dim R$.

REMARK: By the previous proposition, **a local ring $\mathcal{O}_{X,x}$ of a complex variety is regular if and only if x is smooth.**

The Noetherianity is needed because we want to apply the Nakayama lemma as follows.

THEOREM: (Nakayama's lemma)
Let M be a finitely generated module over a Noetherian local ring R, and $\mathfrak{m}$ its maximal ideal. **Then M is generated over R by any collection of elements $m_1, ..., m_n \in M$ which generate $\frac{M}{\mathfrak{m}M}$.**

This statement has a corollary
THEOREM: (Krull lemma)
Let $\mathfrak{m}$ be the maximal ideal in a Noetherian local ring. **Then $\bigcup_i \mathfrak{m}^i = 0$.**

$\mathfrak{m}$-adic topology

DEFINITION: Let G be a group equipped with a left translation-invariant topology. **A Cauchy sequence** is a sequence $\{a_i\}$ such that for any open set $U \subset G$, there exists a left translation L_g such that almost all elements of $\{a_i\}$ belong to $L_g(U)$. A group is called **complete** if all Cauchy sequences converge. **The completion of a group G** is the set of equivalence classes of Cauchy sequences.

DEFINITION: The $\mathfrak{m}$-adic topology on G is topology where open sets are obtained by translations of $\mathfrak{m}^i$ for all $i \in \mathbb{Z}^{>0}$.

EXERCISE: Assume that R is Noetherian. Prove that **the $\mathfrak{m}$-adic completion of a ring R is equal to $\varprojlim R/\mathfrak{m}^i$.**

Claim 1: Let A be a Noetherian local ring, and $I \subset A$ an ideal. **Then I is closed with respect to the adic topology.**

Proof: The closure $\bar{I}$ of I in A is by definition equal to $\bigcap_n (I + \mathfrak{m}^n)$. Then for any $x \in \bar{I}$, its image in A/I belongs to $\bigcap_n \mathfrak{m}^n$. Krull lemma implies that $x \in I$. ■

COROLLARY: In these assumptions, **one has $A \cap (I \cdot \hat{A}) = I$, where $\hat{A}$ is the completion of A.**
Proof: Indeed, $I \cdot \hat{A}$ is contained in the closure of I in $\hat{A}$. ■

Completions of regular local rings are rings of power series

PROPOSITION: Let R be a local ring, $\mathfrak{m}$ its maximal ideal, and $\hat{R}$ the $\mathfrak{m}$-adic completion. Assume that R contains its residue field $R/\mathfrak{m}$ as a subfield. **Then $\hat{R}$ is isomorphic to a formal power series ring $k[[t_1,...,t_n]]$ if and only if R is regular.**

Proof. Step 1: Assume that $\frac{\mathfrak{m}}{\mathfrak{m}^2}$ is n-dimensional, and let $f_1,...,f_n \in \mathfrak{m}$ be elements which generate $\frac{\mathfrak{m}}{\mathfrak{m}^2}$. By Nakayama lemma, $f_1,...,f_n$ generate $\mathfrak{m}$.

Step 2: Denote by $P_d \subset R$ the vector subspace R generated degree d homogeneous polynomials on $f_1,...,f_n$. Clearly, P_d generate $\frac{\mathfrak{m}^d}{\mathfrak{m}^{d+1}}$. Therefore, the natural map $\oplus_{i=0}^n P_i \longrightarrow R/\mathfrak{m}^{n+1}$ is surjective. **We obtained that the image $P_{\{f_i\}}$ of $k[f_1,...,f_n]$ in R is dense in $\mathfrak{m}$-adic topology.**

Step 3: By Krull lemma, R is naturally embedded to $\hat{R}$ **(prove it)**. Since the Krull dimension of a local ring is equal to the degree of its Hilbert polynomial $h_R(n) = \dim_k \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}}$, **we obtain $\dim P_{\{f_i\}} = \dim R = \dim \hat{R}$.**

Step 4: This implies that $\dim k[[t_1,...,t_n]] = n$**, and this ring is regular.**

Completions of regular local rings are rings of power series

Step 5: Suppose that $\hat{R} \cong k[[t_1, ..., t_n]]$. Then

$$\dim \hat{R} = \dim k[[t_1, ..., t_n]] = n = \dim \frac{\mathfrak{m}}{\mathfrak{m}^2}.$$

Since $n = \dim \hat{R} = \dim R$, we obtain that $\dim R = n$ and R is regular.

Step 6: Conversely, assume that R is regular. Since $\hat{R}$ is a completion of $P_{\{f_i\}}$, it would suffice to show that $P_{\{f_i\}}$ is a polynomial algebra. However, $\dim P_{\{f_i\}} = \dim R = n$ (Step 3). If $P_{\{f_i\}}$ is not a polynomial algebra, it is isomorphic to $k[t_1, ..., t_n]/J$, for a non-zero ideal J, and then $\dim P_{\{f_i\}} < n$, contradicting $\dim P_{\{f_i\}} = \dim R = n$. **Then $P_{\{f_i\}}$ is a polynomial algebra, and its completion $\hat{R} = \hat{P}_{\{f_i\}}$ is isomorphic to $k[[t_1, ..., t_n]]$.** ■

Divisibility in the ring of germs

Lemma 1: Let $x \in X$ be a smooth point on an algebraic variety, $\mathcal{O}_{X,x}$ the ring of germs of holomorhic functions, and $\mathcal{O}_{X,x}^{reg}$ the ring of germs of regular functions. Assume that $f \in \mathcal{O}_{X,x}^{reg} \subset \mathcal{O}_{X,x}$ is divisible by $g \in \mathcal{O}_{X,x}^{reg} \subset \mathcal{O}_{X,x}$ in $\mathcal{O}_{X,x}$. **Then f is divisible by g in $\mathcal{O}_{X,x}^{reg}$.**

Proof. Step 1: Since $\mathcal{O}_{X,x}^{reg}$ is dense in $\mathcal{O}_{X,x}$, the function $h := f/g$ is a limit $h = \lim_i h_i$, where $h_i \in \mathcal{O}_{X,x}^{reg}$. Then $f - h_i g \in \mathfrak{m}^i$.

Step 2: Consider the local ring $A := \frac{\mathcal{O}_{X,x}^{reg}}{(g)}$, and let $\mathfrak{n}$ be its maximal ideal. Denote by $\check{f}$ the image of f in this ring. Step 1 gives $f - h_i g \in \mathfrak{m}^i$. This implies that $\check{f} \in \bigcap_i \mathfrak{n}^i$; by Krull's lemma, $\bigcap_i \mathfrak{n}^i = 0$, which implies $f \in (g)$. ■

Proof of Auslander-Buchsbaum theorem

Factoriality of $\mathcal{O}_{X,x}^{reg}$ is immediately implied by the following result.
THEOREM: Let $x \in X$ be a smooth point on an algebraic variety, and $\mathcal{O}_{X,x}^{reg}$ the ring of germs of regular functions. Consider $f, g, h \in \mathcal{O}_{X,x}^{reg}$ such that f divides gh (we denote it as $f|gh$). **Then $f = g_1 h_1$, where $g_1|g$ and $h_1|h$.**

Proof. Step 1: The ring of germs of holomorhic functions $\mathcal{O}_{X,x}$ is factorial (Lecture 5). Using factoriality, we decompose g and h in $\mathcal{O}_{X,x}$ as $g = g'd$, $f = f'd$, where d is their greatest common divisor. Then $fg' = gf'$. Choose, as above, sequences $\{g_i'\}, \{f_i'\} \subset \mathcal{O}_{X,x}^{reg}$ converging to g', f'. **Then $fg_n' - gf_n' = f(g' - g_n') - g(f' - f_n') \in (f,g)\mathfrak{m}^n$, where (f,g) is the ideal generated by (f,g).**

Step 2: The equation $fg_n' - gf_n' \in (f,g)\mathfrak{m}^n$ was proven in $\mathcal{O}_{X,x}$, but in fact it is true in $\mathcal{O}_{X,x}^{reg}$ as well. Claim 1 implies that for any ideal in $I \subset \mathcal{O}_{X,x}^{reg}$, we have $\mathcal{O}_{X,x}^{reg} \cap (I \cdot \mathcal{O}_{X,x}) = I$. Applying this to $I = (f,g)\mathfrak{m}^n$, **we obtain that $fg_n' - gf_n' = -fs_n + gr_n$, where $s_n, r_n \in \mathfrak{m}^n$ are elements of $\mathcal{O}_{X,x}^{reg}$.**

Proof of Auslander-Buchsbaum theorem (2)

Step 3: Replacing g by dg' and and f by df' and dividing by d, the equation $fg'_n - gf'_n = -fs_n + gr_n$ becomes $f'g'_n - g'f'_n = -f's_n + g'r_n$, equivalently, $f'(g'_n + s_n) = g'(r_n + f'_n)$. Since f', g' are coprime, **this implies that $(g'_n + s_n)$ is divisible by g' and $(r_n + f'_n)$ is givisible by f'.**

Step 4: This gives $(r_n + f'_n) = uf'$. Choose n such that $f' \neq 0 \mod \mathfrak{m}^n$. Since $f' = f'_n \mod \mathfrak{m}^n$, the equation $f' = uf' \mod \mathfrak{m}^n$ implies that $u \notin \mathfrak{m}$, hence u is invertible.

Step 5: We have shown that $\frac{f'}{r_n + f'_n}$ is invertible, hence $F' := r_n + f'_n$ divides f. Similarly, $G' := s_n + g'_n$ divides g. Lemma 1 implies that $F'|f$ and $G'|g$ in $\mathcal{O}^{reg}_{X,x}$. The elements G' and F' are coprime in $\mathcal{O}_{X,x}$, because g' and f' are coprime. Let $D \in \mathcal{O}^{reg}_{X,x}$ be an element such that $f = DF'$. From $f|DG'h$ we obtain $F'|G'h$. Since F' is coprime with G', the function F' divides h in $\mathcal{O}_{X,x}$. Applying Lemma 1 again, we see that $f = DF'$, where $D|g$ and $F'|h$ in $\mathcal{O}^{reg}_{X,x}$. ■

REMARK: We used factoriality of $\mathcal{O}_{X,x}$. **However, we could give a purely algebraic proof** if we show that the ring of power series is factorial, and apply the same argument to the completion of $\mathcal{O}^{reg}_{X,x}$, which is isomorphic to $\mathbb{C}[[t_1, ..., t_n]]$, instead of $\mathcal{O}_{X,x} \cong \mathcal{O}_n$.