Complex analytic spaces

lecture 20: Degree of a map and degree of a field extension

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Spectrum of a ring (reminder)

DEFINITION: **Spectrum** of a ring R is the set $\operatorname{Spec} R$ if its prime ideals.

DEFINITION: Let $J \subset R$ be an ideal, and $V(J) \subset \operatorname{Spec} R$ be the set of all prime ideals containing J. **Zariski topology** on $\operatorname{Spec} R$ is topology where all $V(J)$ (and only those) are closed. Clearly, $V(J_1) \cap V(J_2) = V(J_1 + J_2)$ and $V(J_1) \cup V(J_2) = V(J_1 J_2)$, hence finite unions and intersections of closed sets are closed.

DEFINITION: Let R be a ring, $\operatorname{Spec}(R)$ its spectum and $f \in R$. **Affine open set** is an open set $U_f := \operatorname{Spec} R \backslash V_{(f)}$. We identify U_f with $\operatorname{Spec}(R[f^{-1}])$ (localization in f).

EXERCISE: Prove that **finite intersection of affine open sets is affine, $U_f \cap U_g = U_{fg}$.**

EXERCISE: Prove that **affine open sets give a base of Zariski topology.**

Affine schemes (reminder)

DEFINITION: The sheaf $\mathcal{O}$ of regular functions on $\operatorname{Spec} R$ is defined as the sheaf which satisfies $\mathcal{O}\Big|_{U_f} = R[f^{-1}]$, with restriction maps taking a function to its restriction to an open set.

EXERCISE: Prove that $\mathcal{O}\Big|_{U_f} = R[f^{-1}]$ **is sufficient to define a sheaf,** which is reconstructed uniquely from this property.

DEFINITION: A scheme is a ringed space $(M, \mathcal{O})$, which is locally isomorphic to an affine scheme with the sheaf of regular functions. In this situation sheaf $\mathcal{O}$ is called **the structure sheaf** of the scheme, or **the sheaf of regular functions**.

REMARK: The structure sheaf **may contain nilpotents. An algebraic variety** is a scheme which does not have nilpotents in its structure sheaf.

DEFINITION: Morphism of affine schemes is a morphism of ringed spaces $\operatorname{Spec} A \longrightarrow \operatorname{Spec} B$ induced by a ring homomorphism $B \longrightarrow A$. **Morphism of schemes** is a map of schemes which is given by morphisms of affine schemes in local affine charts.

Degree of a map and degree of the field extension

The main result of today's lecture:

THEOREM: Let $\varphi : X \longrightarrow Y$ be a dominant regular map of n-dimensional irreducible complex algebraic varieties, and $k(X), k(Y)$ their rational function fields. Let d be the degree of the field extension $[k(X) : k(Y)]$ **Then there exists a Zariski open subset $Y_0 \subset Y$ such that each $y \in Y_0$ has precisely d preimages.**

Proof. Step 1: Replacing X, Y by Zariski open subsets, we can always assume that X, Y are affine subvarieties in $\mathbb{C}^n$. Let $X_1, ..., X_n$ be coordinate functions on X. By the primitive element theorem, **an appropriate linear combination $u = \sum \alpha_i x_i$ generates $k(X)$ over $k(Y)$.**

Step 2: Let $\Gamma \subset Y \times \mathbb{C}$ be the Zariski closure of the image of $\varphi \times u$. We decompose φ into a composition of $X \stackrel{\varphi\times u}{\longrightarrow} \Gamma$ and the projection $\Gamma \longrightarrow Y$. **Since u generates $k(X)$ over $k(Y)$, we have $k(\Gamma) = k(X)$.**

Degree of a map and degree of the field extension (2)

Step 3: It remains to show that the projection $\Gamma \longrightarrow Y$ is a degree d ramified covering, in other words, a general point in Y has $[k(X) : k(Y)]$ preimages in Γ. Since $\dim \Gamma = \dim Y$, the field $k(\Gamma)$ has the same transcendence degree as $k(Y)$. Therefore, the coordinate u satisfies a polynomial equation $P(u) = \sum_{i=0}^r u^i a_i = 0$, where a_i are regular functions on Y. **Since Γ is irreducible, the polynomial $P(u)$ is irreducible over $k(Y)$;** otherwise we would have several components over the general point, and their closure would give several irreducible components for Γ. **This polynomial has degree d, because $k(X) = k(\Gamma) = k(Y)[u]$.**

Step 4: The fiber of the projection $\Gamma \longrightarrow Y$ is the set of all solutions of $P(u) = 0$. Since $P(u)$ is irreducible, its discriminant is non-zero, and **outside of its discriminant, there are precisely d different solutions.** ■