Complex analytic spaces

lecture 22: The integral closure

Misha Verbitsky

IMPA, sala 236,

October 18, 2023, 17:00

Spectrum of a ring (reminder)

DEFINITION: Spectrum of a ring R is the set $\operatorname{Spec} R$ if its prime ideals.

DEFINITION: Let $J \subset R$ be an ideal, and $V(J) \subset \operatorname{Spec} R$ be the set of all prime ideals containing J. **Zariski topology** on $\operatorname{Spec} R$ is topology where all $V(J)$ (and only those) are closed. Clearly, $V(J_1) \cap V(J_2) = V(J_1 + J_2)$ and $V(J_1) \cup V(J_2) = V(J_1 J_2)$, hence finite unions and intersections of closed sets are closed.

DEFINITION: Let R be a ring, Spec(R) its spectum and $f \in R$. **Affine open set** is an open set $U_f := \operatorname{Spec} R \backslash V_{(f)}$. We identify U_f with $\operatorname{Spec}(R[f^{-1}])$ (localization in f).

EXERCISE: Prove that **finite intersection of affine open sets is affine, $U_f \cap U_g = U_{fg}$.**

EXERCISE: Prove that **affine open sets give a base of Zariski topology.**

Affine schemes (reminder)

DEFINITION: The sheaf $\mathcal{O}$ of regular functions on $\operatorname{Spec} R$ is defined as the sheaf which satisfies $\mathcal{O}\big|_{U_f} = R[f^{-1}]$, with restriction maps taking a function to its restriction to an open set.

EXERCISE: Prove that $\mathcal{O}\big|_{U_f} = R[f^{-1}]$ **is sufficient to define a sheaf,** which is reconstructed uniquely from this property.

DEFINITION: A scheme is a ringed space $(M, \mathcal{O})$, which is locally isomorphic to an affine scheme with the sheaf of regular functions. In this situation sheaf $\mathcal{O}$ is called **the structure sheaf** of the scheme, or **the sheaf of regular functions**.

REMARK: The structure sheaf **may contain nilpotents. An algebraic variety** is a scheme which does not have nilpotents in its structure sheaf.

DEFINITION: Morphism of affine schemes is a morphism of ringed spaces $\operatorname{Spec} A \longrightarrow \operatorname{Spec} B$ induced by a ring homomorphism $B \longrightarrow A$. **Morphism of schemes** is a map of schemes which is given by morphisms of affine schemes in local affine charts.

Dominant morphisms (reminder)

DEFINITION: Zariski topology on an algebraic variety is a topology where the closed subsets are algebraic subsets. **Zariski closure** of $Z \subset M$ is an intersection of all Zariski closed subsets containing Z.

EXERCISE: Prove that **Zariski topology on $\mathbb{C}$ coincides with the cofinite topology.**

CAUTION: Zariski topology is non-Hausdorff.

DEFINITION: Dominant morphism is a morphism $f: X \longrightarrow Y$, such that Y is a Zariski closure of $f(X)$.

PROPOSITION: Let $f: X \longrightarrow Y$ be a morphism of affine varieties. **The morphism f is dominant if and only if the homomorphism $\mathcal{O}_Y \xrightarrow{f^*} \mathcal{O}_X$ is injective**.

Field of fractions (reminder)

DEFINITION: Let $S \subset R$ be a subset of R, closed under multiplication and not containing 0. **Localization** of R in S is a ring, formally generated by symbols a/F, where $a \in R$, $F \in S$, and relations $a/F \cdot b/G = ab/FG$, $a/F + b/G = \frac{aG+bF}{FG}$ and $aF^k/F^{k+n} = a/F^n$.

DEFINITION: Let R be a ring without zero divisors, and S the set of all non-zero elements in R. **Field of fractions** of R is a localization of R in S.

CLAIM: Let $f: X \longrightarrow Y$ be a dominant morphism, where X is irreducible. **Then Y is also irreducible**. Moreover, $f^*: \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ **can be extended to a homomorphism of the fields of fractions. $k(Y) \longrightarrow k(X)$.**

DEFINITION: A dominant morphism of irreducible varieties is called **birational** if the corresponding homomorphism of the fields of fractions is an isomorphism.

Integral dependence

DEFINITION: Let $A \subset B$ be rings. An element $b \in B$ is called **integral over A** if the subring $A[b] = A\cdot\langle 1, b, b^2, b^3, ...\rangle$, generated by b and A, is finitely generated as A-module.

DEFINITION: **Monic polynomial** is a polynomial with leading coefficient 1.

CLAIM: An element $x \in B$ **is integral over $A \subset B$ if and only if the chain of submodules**

$$A \subset A\cdot\langle 1, x\rangle \subset A\cdot\langle 1, x, x^2\rangle \subset A\cdot\langle 1, x, x^2, x^3\rangle \subset ...$$

terminates.

COROLLARY: An element $x \in B$ is integral over $A \subset B$ $\Leftrightarrow$ x is a root of a monic polynomial with coefficients in A. ■

CLAIM: Let $A \subset B$ be Noetherian rings. Then **sum and product of elements which are integral over A is also integral.**

Integral closure

DEFINITION: Let $A \subset B$ be rings. The set of all elements in B which are integral over A is called **the integral closure of A in B**.

DEFINITION: Let A be the ring without zero divisors, and $k(A)$ its field of fractions. The set of all elements $a \in k(A)$ which are integral over A is called **the integral closure of A**. A ring A is called **integrally closed** if A coincides with its interal closure in $k(A)$.

REMARK: As shown above, **the integral closure is a ring.**

DEFINITION: An affine variety X is called **normal** if all its irreducible components X_i are disconnected, and the ring of functions $\mathcal{O}_{X_i}$ for each of these irreducible components is integrally closed.

REMARK: Equivalently, **X is normal if any finite, birational morphism $Y \longrightarrow X$ is an isomorphism.**

Factorial rings

DEFINITION: An element p of a ring R is called **prime** if the corresponding principal ideal (p) is prime.

DEFINITION: A ring R without zero divisors is called **factorial** if any element $r \in R$ can be represented as a product of prime elements, $r = \prod_i p_i^{\alpha_i}$, and this decomposition is unique up to invertible factors and permutation of p_i.

PROPOSITION: Let A be a factorial ring. Then it is integrally closed.

Proof. Step 1: Let $u, v \in A$, and $u/v \in k(A)$ a root of a monic polynomial $P(t) \in A[t]$ of degree n. **Then u^n is divisible by v in A.**

Step 2: Let $u/v \in k(A)$ be a root of a monic polynomial $P(t) \in A[t]$. Assume that u, v are coprime. **Since u^n is divisible by v, and they are coprime, v is invertible by factoriality of A.** Then $u/v \in A$. ∎