Complex analytic spaces

lecture 23: Normalization of complex algebraic varieties

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Spectrum of a ring (reminder)

DEFINITION: Spectrum of a ring R is the set $\operatorname{Spec} R$ if its prime ideals.

DEFINITION: Let $J \subset R$ be an ideal, and $V(J) \subset \operatorname{Spec} R$ be the set of all prime ideals containing J. **Zariski topology** on $\operatorname{Spec} R$ is topology where all $V(J)$ (and only those) are closed. Clearly, $V(J_1) \cap V(J_2) = V(J_1 + J_2)$ and $V(J_1) \cup V(J_2) = V(J_1 J_2)$, hence finite unions and intersections of closed sets are closed.

DEFINITION: Let R be a ring, Spec(R) its spectum and $f \in R$. **Affine open set** is an open set $U_f := \operatorname{Spec} R \backslash V_{(f)}$. We identify U_f with Spec$(R[f^{-1}])$ (localization in f).

EXERCISE: Prove that **finite intersection of affine open sets is affine, $U_f \cap U_g = U_{fg}$.**

EXERCISE: Prove that **affine open sets give a base of Zariski topology.**

Affine schemes (reminder)

DEFINITION: The sheaf $\mathcal{O}$ of regular functions on Spec R is defined as the sheaf which satisfies $\mathcal{O}\big|_{U_f} = R[f^{-1}]$, with restriction maps taking a function to its restriction to an open set.

EXERCISE: Prove that $\mathcal{O}\big|_{U_f} = R[f^{-1}]$ **is sufficient to define a sheaf,** which is reconstructed uniquely from this property.

DEFINITION: A scheme is a ringed space $(M, \mathcal{O})$, which is locally isomorphic to an affine scheme with the sheaf of regular functions. In this situation sheaf $\mathcal{O}$ is called **the structure sheaf** of the scheme, or **the sheaf of regular functions**.

REMARK: The structure sheaf **may contain nilpotents. An algebraic variety** is a scheme which does not have nilpotents in its structure sheaf.

DEFINITION: Morphism of affine schemes is a morphism of ringed spaces Spec $A \longrightarrow$ Spec B induced by a ring homomorphism $B \longrightarrow A$. **Morphism of schemes** is a map of schemes which is given by morphisms of affine schemes in local affine charts.

Integral closure (reminder)

DEFINITION: Let $A \subset B$ be rings. The set of all elements in B which are integral over A is called **the integral closure of A in B**.

DEFINITION: Let A be the ring without zero divisors, and $k(A)$ its field of fractions. The set of all elements $a \in k(A)$ which are integral over A is called **the integral closure of A**. A ring A is called **integrally closed** if A coincides with its interal closure in $k(A)$.

REMARK: As shown above, **the integral closure is a ring.**

DEFINITION: An affine variety X is called **normal** if all its irreducible components X_i are disconnected, and the ring of functions $\mathcal{O}_{X_i}$ for each of these irreducible components is integrally closed.

REMARK: Equivalently, **X is normal if any finite, birational morphism $Y \longrightarrow X$ is an isomorphism.**

PROPOSITION: Let A be a factorial ring. Then it is integrally closed.

Characteristic polynomial over a ring

DEFINITION: Let A be a Noetherian ring, $[K:k(A)]$ a finite extension of its field of fractions, and B the integral closure of A in K. For any $b \in B$, denote by $L_b : K \longrightarrow K$ the map of multiplication by b. Consider L_b as a $k(A)$-linear endomorphism of the finite-dimensional space K over $k(A)$, and define **the trace** of b as $\operatorname{Tr}(b) := \operatorname{Tr}(L_b)$.

CLAIM: In these assumptions, $\operatorname{Tr}(b)$ **is an element of A.**

Proof. Step 1: The coefficients a_i of the minimal polynomial of $b \in B$ over $k(A)$ belong to A. Indeed, let $\tilde{B}$ be the integral closure of $k(A)$ in its algebraic closure. **Then $a_i \in A$ are obtained as elementary polynomials on Galois conjugates of b,** which are all in $\tilde{B}$. Since A is integrally closed, we have $k(A)A \cap \tilde{B} = A$, which implies that $a_i \in A$.

Step 2: Let $B' \subset B$ be the algebra generated by b. Since b is finite, B' is a finitely-generated A-module, isomorphic to $A[t]/P(t)$, where $P(t) \in k(A)[t]$ is the minimal polynomial of P. Since A is integrally closed, the coefficients of $P(t)$ are elements of A. Let $1, b, b^2, ..., b^n$ be its generators over A, and (a_{ij}) the matrix of L_b written in this basis. **Then B is decomposed as a $k(A)$-vector space onto a sum of of d copies of B', and on each of them L_b acts as**

$$(a_{ij}) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix}$$

where $P(t) = t^n + \sum_{i=1}^n a_{n-i} t^{n-i}$. This gives $\operatorname{Tr}(b) = da_{n-1} \in A$. ■

Finiteness of integral closure

THEOREM: Let A be an integrally closed Noetherian ring, $[K:k(A)]$ a finite extension of its field of fractions, and B the integral closure of A in K. **Then B is finitely generated as an A-module.**

Proof. Step 1: The bilinear symmetric form $x, y \longrightarrow \operatorname{Tr}(xy)$ is non-degenerate. Indeed, $\operatorname{Tr}(xx^{-1}) = \dim_{k(A)} K$, and $\operatorname{char} k(A) = 0$.

Step 2: Choose a basis $e_1, ..., e_n$ in the $k(A)$-vector space K. Let $P_i(t) \in k(A)[t]$ be the minimal polynomials of e_i. Write $P_i(t) = A_i t^{n_i} + \sum_{j<n_i} a_{ij} t^j$, where $A_i, a_{ij} \in A$. Then $A_i e_i$ is a root of a monic polynomial $\tilde{P}_i(t) = t^{n_i} + \sum_{j<n_i} A^{n_i - j} a_{ij} t^j$. **This proves that the basis $e_1, ..., e_n$ in $K:k(A)$ can be chosen such that all e_i are integral over A.**

Step 3: Let $e_i^* \in K$ be the dual basis with respect to the form Tr, with $\operatorname{Tr}(e_i^* e_j) = \delta_{ij}$. Consider the A-module $M \subset K$ generated by e_i^*. Clearly, $M := \{b \in K \mid \operatorname{Tr}(be_i) \in A\}$.

Step 4: For any $b \in B$, the trace $\operatorname{Tr}(b)$ belongs to A, because b is integral over A (Step 1). Then $B \subset M$, and B **is a submodule of a finitely generated A-module M.** Since A is Noetherian, B is finitely generated as A-module. ■

Finiteness of integral closure: first applications

COROLLARY: Let B be a ring over $\mathbb{C}$. Assume that there exists an injective ring morphism from $A = \mathbb{C}[x_1, ..., x_k]$ to B such that B is finitely generated as an A-module. **Then its integral closure $\hat{B}$ is a finitely generated A-module**. In particlular, $\hat{B}$ **is a finitely generated ring.**

Proof: Since A is factorial, it is integrally closed, and the previous theorem applies. ■

DEFINITION: Let X be an affine variety, and $\hat{A}$ the integral closure of its ring of regular functions. Assume that $\hat{A}$ is a finitely generated ring. Then $\hat{X} := \operatorname{Spec}(\hat{A})$ is called **the normalization of X**.

REMARK: Today we shall prove that $\hat{A}$ **is always finitely generated,** if A is a finitely generated ring over $\mathbb{C}$.

Transcendence basis

DEFINITION: Let $k(t_1, ..., t_n)$ be the field of rational functions of several variables, that is, the fraction field for the polynomial ring $k[t_1, ..., t_n]$. Then the extension $[k(t_1, ..., t_n) : k]$ is called **a purely transcendental extension of** k, and $t_1, ..., t_n$ are called **algebraically independent**.

REMARK: Clearly, $t_1, ..., t_n$ **are algebraically independent if and only if there are no alrebraic relations of form** $P(t_1, ..., t_n) = 0$**,** where P is a polynomial of n variables.

DEFINITION: Transcendence basis of an extension $[K : k]$ is a collection $z_1, ..., z_n \in K$ generating a purely trascendental extension $K' := k(z_1, ..., z_n)$ such that $[K : K']$ is an algebraic extension. We call the number n **the transcendental degree** of X.

CLAIM: Let $X \subset \mathbb{C}^n$ be an irreducible affine manifold, $t_1, ..., t_n$ coordinates on $\mathbb{C}^n$, and $\Pi_k : X \longrightarrow \mathbb{C}^k$ the projection to the first k coordinates. **Then the following are equivalent.**

(i) Π_k **is dominant and the extension** $[k(X) : k(t_1, ..., t_k)]$ **is finite.**

(ii) $t_1, ..., t_k$ **is transcendence basis in** $k(X)$**.** ■

When the coordinate projection is finite

REMARK: Let $X \subset \mathbb{C}^n$ be an irreducible affine subvariety, z_i coordinates on $\mathbb{C}^n$, and $z_1, ..., z_k$ transcendence basis on $k(X)$. **The projection map Π_{n-1} is finite if and only if $P(z_n) = 0$ in $\mathcal{O}_X$, for some monic polynomial $P(t) \in \mathcal{O}_X[t]$ with coefficients which are polynomial in $z_1, ..., z_{n-1}$.** Indeed, this is precisely what is needed for $\mathcal{O}_X$ to be a finitely generated module over its subalgebra $A = \mathbb{C}[z_1, ..., z_{n-1}]$. Notice that **a non-zero polynomial $P(t) \in A[t]$ such that $P(z_n) = 0$ on X always exists, unless $n = k$ and $X = \mathbb{C}^n$,** but **it is not necessarily monic.**

CLAIM: In these assumptions, **there exists a linear coordinate change $z_i' := z_i + \lambda_i z_n$, such that z_n is finite over $z_1', ..., z_k'$.**

Proof: Next slide.

REMARK: This immediately implies **the Noether's normalization lemma**: **any affine manifold admits a finite, dominant map to $\mathbb{C}^n$.**

When the coordinate projection is finite (2)

CLAIM: In these assumptions, **there exists a linear coordinate change $z_i' := z_i + \lambda_i z_n$, such that z_n is finite over $z_1', ..., z_k'$.**

Proof. Step 1: Let $P(z_1, ..., z_k, t)$ be a non-zero polynomial such that $P(z_1, ..., z_k, z_n) = 0$ in $\mathcal{O}_X$. Such a polynomial exists because $z_1, ..., z_k$ is a transcendence basis in $k(X)$, and z_n is algebraic over $z_1, ..., z_k \in \mathcal{O}_X$. Let $F(z_1, ..., z_k, z_n)$ be a homogeneous component of maximal degree in $P(z_1, ..., z_k, z_n)$. We choose P to be of minimal possible degree in $z_1, ..., z_k, z_n$.

Step 2: Consider a polynomial

$$Q(z_1, ..., z_k, z_n) := F(z_1 + \lambda_1 z_n, ..., z_k + \lambda_k z_n, z_n).$$

Then $Q(0, 0, ..., 0, 1) = F(\lambda_1, ..., \lambda_k, 1)$ is non-zero for general λ_i. Indeed, if $F(\lambda_1, ..., \lambda_k, 1)$ is identically 0 for all λ_i, the homogeneous polynomial F vanishes.

Step 3: Let $z_i' := z_i + \lambda_i z_n$. The degree d polynomial $P(z_1, ..., z_k, z_n) = Q(z_1', ..., z_k', z_n)$ is monic in z_n, because its leading term z_n^d has non-zero coefficient by Step 2. ∎

Normalization of an affine variety

COROLLARY: Let X be an affine variety, and A the integral closure of its ring of regular functions. **Then A is finitely generated.**

Proof: Let $X \longrightarrow \mathbb{C}^d$ be a finite, dominant map. Since all elements of $\mathcal{O}_X$ are finite over $\mathcal{O}_{\mathbb{C}^d}$, the ring $\mathcal{O}_X$ is contained in the integral closure $\hat{A}$ of $\mathcal{O}_{\mathbb{C}^d}$ in $k(\mathcal{O}_X)$. This implies that $A \subset \hat{A}$. On the other hand, all elements of $\hat{A}$ are finite over $\mathcal{O}_{\mathbb{C}^d}$, hence they are finite over $\mathcal{O}_X$, which implies $\hat{A} \subset A$. ■

DEFINITION: Let X be an affine variety, and $\hat{A}$ the integral closure of its ring of regular functions. Then $\tilde{X} := \operatorname{Spec}(\hat{A})$ is called **normalization of X**.

REMARK: The normalization map is finite and birational; **X is normal if for any finite, birational $\varphi : X' \longrightarrow X$, the map φ is an isomorphism**. Indeed, in this case $\mathcal{O}_{X'} \supset \mathcal{O}_X$ is finite with the same field of fractions.

COROLLARY: Normalization of X is a finite, birational morphism $X' \longrightarrow X$ **such that for any other finite, birational $\varphi : X'' \longrightarrow X'$, the map φ is an isomorphism**. In particular, **any birational, finite map $X' \longrightarrow X$ with X' normal is a normalization.** ■

Preimage and diagonal

Claim 2: Let $f: X \longrightarrow Y$ be a morphism of algebraic varieties, $f^*: \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ the corresponding ring homomorphism, $Z \subset Y$ a subvariety, and I_Z its ideal. Denote by R_1 the quotient of a ring $R := \mathcal{O}_X \otimes_{\mathcal{O}Y} (\mathcal{O}_Y/I_Z) = \mathcal{O}_X/f^*(I_Z)$ by its nilradical. **Then** $\operatorname{Spec}(R_1) = f^{-1}(Z)$**.**

Proof: Clearly, the set of common zeros of the ideal $J := f^*(I_Z)$ contains $f^{-1}(Z)$. On the other hand, for any point $x \in X$ such that $f(x) \notin Z$ there exist a function $g \in J$ such that $g(x) \neq 0$. Therefore, $f^{-1}(Z) = V_J$, and strong Nullstellensatz implies that $\mathcal{O}_{f^{-1}(Z)} = R_1$. ■

Claim 3: Let M be an algebraic variety, $\Delta \subset M \times M$ the diagonal, and $I \subset \mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{O}_M$ the ideal generated by $r \otimes 1 - 1 \otimes r$ for all $r \in \mathcal{O}_M$. **Then** $\mathcal{O}_\Delta$ **is** $\mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{O}_M/I$**.**

Proof. Step 1: By definition of the tensor product, $\mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{O}_M/I = \mathcal{O}_M \otimes_{\mathcal{O}_M} \mathcal{O}_M = \mathcal{O}_M$, hence it is reduced. If we prove that $\Delta = V_I$, the statement of the claim would follow from strong Nullstellensatz.

Step 2: Clearly, $\Delta \subset V_I$. To prove the converse, let $(m, m') \in M \times M$ be a point not on diagonal, and $f \in \mathcal{O}_M$ a function which satisfies $f(m) = 0, f(m') \neq 0$. Then $f \otimes 1 - 1 \otimes f$ is non-zero on (m, m'). ■

Fibered product

DEFINITION: Let $X \xrightarrow{\pi_X} M, Y \xrightarrow{\pi_Y} M$ be maps of sets. **Fibered product** $X \times_M Y$ is the set of all pairs $(x,y) \in X \times Y$ such that $\pi_X(x) = \pi_Y(y)$.

CLAIM: Let $X \xrightarrow{\pi_X} M, Y \xrightarrow{\pi_Y} M$ be morphism of algebraic varieties, $R := \mathcal{O}_X \otimes_{\mathcal{O}_M} \mathcal{O}_Y$, and R_1 the quotient of R by its nilradical. **Then** Spec$(R_1) = X \times_M Y$.

Proof: Let I be the ideal of diagonal in $\mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{O}_M$. Since I is generated by $r \otimes 1 - 1 \otimes r$ (Claim 3), $R = \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_Y / (\pi_X \times \pi_Y)^*(I)$. Applying Claim 2, we obtain that Spec$(R_1) = (\pi_X \times \pi_Y)^{-1}(\Delta)$. ■

Functoriality of normalization

CLAIM: Let X be an affine variety, $\hat{X}$ its normalization, and $U \subset X$ a Zariski open subvariety. **Then the normalization of U can be obtained as** Spec$(\mathcal{O}_U \otimes_{\mathcal{O}_X} \mathcal{O}_{\hat{X}})$.

Proof. Step 1: Indeed, any element $f \in \mathcal{O}_U \otimes_{\mathcal{O}_X} \mathcal{O}_{\hat{X}}$ is expressed as $f = \sum a_i f_i$, where $a_i \in \mathcal{O}_U$ and $f_i \in \mathcal{O}_{\hat{X}}$ are finite over $\mathcal{O}_U$, hence f is finite over $\mathcal{O}_U$.

Step 2: Conversely, any element f which is finite over $\mathcal{O}_U$ satisfies an equation $P(t) = 0$, where $P(t)$ is a monic polynomial in $\mathcal{O}_U(t)$. Then there exists $u \in \mathcal{O}_X$, invertible in $\mathcal{O}_U$, such that $uP(t) \in \mathcal{O}_X[t]$. Let $uP(t) = ut^n + \sum_{i=1}^{n-1} a_i t^{n-i}$, where $a_i \in \mathcal{O}_X$. **Then uf satisfies an equation $\frac{t^n}{u^{n-1}} + \sum_{i=1}^{n-1} a_i (t/u)^{n-i}$, equivalently, $t^n + \sum_{i=1}^{n-1} a_i u^{i-1} t^{n-i}$ hence $uf \in \mathcal{O}_{\hat{X}}$.** This gives $f \in \mathcal{O}_U \otimes_{\mathcal{O}_X} \mathcal{O}_{\hat{X}}$. ■

REMARK: The intersection of open subsets is their fibered product. Since Spec$(\mathcal{O}_U \otimes_{\mathcal{O}_X} \mathcal{O}_{\hat{X}}) = U \times_X \hat{X}$, the previous claim can be expressed as $\hat{U} = U \times_X \hat{X}$. Similarly, for two open set $U, W \subset X$, one has $\widehat{U \cap W} = \hat{U} \times_X \hat{W}$.

Normalization of a scheme

REMARK: A scheme can be defined as a collection of affine charts $\{U_i\}$ together with the open subvarieties $U_{ij} \subset U_i$ and $U_{ji} \subset U_j$ and **gluing maps**, isomorphisms $\psi_{ij} : U_{ij} \longrightarrow U_{ji}$ which satisfy the **cocycle condition**: for any triple of indices i, j, k, the restriction of $\psi_{ij} \circ \psi_{jk}$ to $U_{ij} \cap U_{ik}$ is equal to ψ_{ik}.

DEFINITION: Let X be a reduced, irreducible scheme, $\{U_i\}$ its affine covering, and $\hat{U}_i$ the normalizations of every affine scheme U_i. Using the previous claim, we obtain that $(\widehat{U_i \cap U_j}) = \hat{U}_i \times_X \hat{U}_j$. **This implies that the gluing map between the affine sets $\{\hat{U}_i\}$ satisfy the cocycle condition, and these affine sets can be glued together to a scheme.** This scheme is called **the normalization of X**.