Complex analytic spaces

lecture 24: Normalization

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Integral closure (reminder)

DEFINITION: Let $A \subset B$ be rings. The set of all elements in B which are integral over A is called **the integral closure of A in B**. The set of all elements $a \in k(A)$ in the field of fractions of A which are integral over A is called **the integral closure of A**. A ring A is called **integrally closed** if A coincides with its integral closure in $k(A)$.

REMARK: As shown in Lecture 22, **the integral closure is a ring.**

PROPOSITION: **Let A be a factorial ring. Then it is integrally closed.**

THEOREM: Let A be an integrally closed Noetherian ring, $[K : k(A)]$ a finite extension of its field of fractions, and B the integral closure of A in K. **Then B is finitely generated as an A-module**.

Corollary 1: Let B be a ring over $\mathbb{C}$. Assume that there exists an injective ring morphism from $A = \mathcal{O}_n$ to B such that B is finitely generated as an A-module. **Then its integral closure $\hat{B}$ is a finitely generated A-module**. In particlular, **$\hat{B}$ is a finitely generated ring.**

Proof: Since A is factorial, it is integrally closed, and the previous theorem applies. ∎

Normal complex varieties

DEFINITION: A complex variety X is called **normal** if for any $x \in X$ the corresponding ring of germs $k(\mathcal{O}_{X,x})$ is an integrally closed ring without zero divisors.

EXAMPLE: All smooth varieties are normal. Indeed, the ring $\mathcal{O}_n$ is factorial, hence integrally closed.

CLAIM: Let X be a normal variety, and U an irreducible open set. **Then the ring $H^0(\mathcal{O}_U)$ of holomorphic functions is integrally closed.**

Proof: Let $f \in k(H^0(\mathcal{O}_U))$ be a meromorphic function which is finite over the ring $H^0(\mathcal{O}_U)$. Then each of its germs in $z \in U$ is finite over $\mathcal{O}_{X,z}$, hence belongs to $\mathcal{O}_{X,z}$. Then all germs of f are holomorphic, and this function is therefore holomorphic. ■

Normal complex varieties (2)

CLAIM: Conversely, **if $H^0(\mathcal{O}_U)$ is integrally closed for all open irreducible $U \subset X$, the variety X is normal.**

Proof: Let $f \in k(\mathcal{O}_{X,x})$ be a function which is finite over $\mathcal{O}_{X,x}$; then f is a root of a monic polynomial $P(t) = 0$ with coefficients in $\mathcal{O}_{X,x}$. Let V be an open neighbourhood of x such that $f \in k(H^0(\mathcal{O}_V))$. Each of the coefficients of $P(t)$ is defined in some open set containing x, **hence $P(t)$ has coefficients in $H^0(\mathcal{O}_W)$, where W is the intersection of V and all these open sets.** Since $H^0(\mathcal{O}_W)$ is integrally closed, we have $f \in \mathcal{O}_W$, hence $f \in \mathcal{O}_{X,x}$. ■

Divisors in normal varieties

PROPOSITION: Let X be a normal complex variety, and $D \subset X$ a divisor. **Then there exists a proper subvariety $D_1 \subset D$ such that in all points $x \in D \backslash D_1$, the ideal of D is principal in $\mathcal{O}_{X,x}$.**

Proof. Step 1: Let R be the localization of $\mathcal{O}_{X,x}$ in I_D, a and $\mathfrak{m} = I_D R$ its maximal ideal. To finish the proof **it would suffice to show that $\mathfrak{m}$ is a principal ideal in R.** Indeed, in this case, the ideal I_D is generated by some $v \in I_D$ on a complement to a collection of divisors which intersect D in a proper subvariety.

Step 2: Removing subsets of codimension $\geqslant 2$, we we may always assume that D is irreducible and $x \in D$ a smooth point in D. Let $f \in \mathcal{O}_{X,x}$ be a non-zero function which vanishes on D. **By Rückert Nullstellensatz, the ideal I_D of D is the radical $\sqrt{(f)}$ of the principal ideal (f).**

Divisors in normal varieties (2)

Steps 1-2: Let R be the localization of $\mathcal{O}_{X,x}$ in I_D, a and $\mathfrak{m} = I_D R$ its maximal ideal. To finish the proof **it would suffice to show that $\mathfrak{m}$ is a principal ideal in R.** Let $f \in \mathcal{O}_{X,x}$ be a non-zero function which vanishes on D. **By Rückert Nullstellensatz, the ideal I_D of D is the radical $\sqrt{(f)}$ of the principal ideal (f).**

Step 3: Let k be the minimal number such that $\mathfrak{m}^k \subset (f)$. Then for some $\alpha_1, ..., \alpha_{k-1} \in \mathfrak{m}$, we have $\alpha_1\alpha_2...\alpha_{k-1}\mathfrak{m} \subset (f)$ and $\alpha_1\alpha_2...\alpha_{k-1} \notin (f)$. Let $g = \alpha_1\alpha_2...\alpha_{k-1}$, and $u := \frac{g}{f}$. **Then $u\mathfrak{m} \subset R$, and $u \notin R$.**

Step 4: If $u\mathfrak{m} \subset \mathfrak{m}$, consider the subalgebra $A \subset \operatorname{Hom}_R(\mathfrak{m}, \mathfrak{m})$ generated by u. This algebra is a finitely generated as a module over R, hence any element of A is a root of a monic polynomial. **Since R is integrally closed, this implies that $u \in R$, a contradiction.**

Step 5: Since $u\mathfrak{m} \not\subset \mathfrak{m}$, this implies that $u\mathfrak{m} = R$, hence $u^{-1}R = \mathfrak{m}$, and the ideal $\mathfrak{m}$ is principal. ■

Normal varieties are smooth in codimension 1

THEOREM: Let X be a normal complex variety, and X_{sing} the set of its singular points (it is complex analytic, as shown in Lecture 12). **Then** $\operatorname{codim} X_{\text{sing}} \geqslant 2$**.**

Proof. Step 1: Suppose, by absurd, that $\operatorname{codim} X_{\text{sing}} = 1$. After removing a subset of $\operatorname{codim}_X \geqslant 2$, we may assume that $D = X_{\text{sing}}$ is a divisor. Using the previous proposition, **we may also assume that the corresponding ideal is principal.**

Step 2: Recall that a local ring of Krull dimension d **is regular if its maximal ideal can be generated by d elements** (Lecture 19). Let $n = \dim X$. Let $x \in D$ be a smooth point in D. **To prove the theorem it would suffice to show that x is smooth in X,** that is, **to prove that the maximal ideal $\mathfrak{m}$ of x in $\mathcal{O}_{X,x}$ is generated by n elements**.

Step 3: Let f be the generator of I_D in $\mathcal{O}_{X,x}$, and $u_1, ..., u_{n-1} \in \mathcal{O}_{X,x}/I_D = \mathcal{O}_D$ functions generating the maximal ideal of $x \in D$. Such functions exist, because $x \in D$ is smooth. **Then $\mathfrak{m}$ is generated by f and $u_1, ..., u_{n-1}$,** implying that $x \in X$ is smooth. ■

COROLLARY: Let X be a normal complex curve. **Then X is smooth.** ■

Normalization

DEFINITION: We say that a complex variety X **admits a normalization** if there exists a finite, bimeromorphic holomorphic map $F: \tilde{X} \longrightarrow X$; in this case $\tilde{X}$ is called **the normalization** of X, and F **the normalization map.**

PROPOSITION: Let X be an irreducible complex variety. **Then every point $x \in X$ has an open neighbourhood which admits a normalization.**

Proof: By finiteness theorem, a germ (X,x) of a complex variety admits a finite, dominant holomorphic map to $\mathbb{C}^d$. By Corollary 1, the integral closure of $\mathcal{O}_{X,x}$ in $k(\mathcal{O}_{X,x})$ is finitely generated. Let $z_1, ..., z_m \in k(\mathcal{O}_{X,x})$ be the generators of the integral closure, $P_1, ..., P_r \in \mathcal{O}_{X,x}[z_1, ..., z_m]$ the generators of the ideal of all relations between these generators, and $U_x \ni x$ an open neighbourhood of x where all coefficients of these relations are holomorphic. Then **the normalization $\tilde{U}_x$ of U_x is a subvariety in $X \times \mathbb{C}^m$ defined by the relations $P_1, ..., P_r$.** ■

CLAIM: The normalization is unique up to an isomorphism.

Proof: Indeed, the holomorphic functions on $\tilde{X}$ are meromorphic functions on X which are finite over $\mathcal{O}_X$. Then for every two normalizations, their ring of functions are isomorphic, which defines the correspondende between the coordinate functions, and this defines a biholomorphic equivalence. ■

Every complex variety admits a normalization

THEOREM: Every complex variety admits a normalization.

Proof: Let X be a complex variety, and $\{U_i\}$ its covering by open sets admitting a normalization. A restriction of the normalization to a smaller open subset is again a normalization, because normality is a local property. Take a variety with an atlas $\{\tilde{U}_i\}$ and the gluing maps obtained by lifting the gluing maps of $U_i \cap U_j$ to the normalization. The cocycle condition is automatic, because if a relation is true on $\mathcal{O}_{U_i \cap U_j \cap U_k}$, it is also true on its integral closure. Then the atlas $\{\tilde{U}_i\}$ with these gluing maps defines a variety $\tilde{X} \longrightarrow X$, which is normal, because normality is a local property. ■

REMARK: The normalization of a complex curve is its desingularization.