Complex analytic spaces

lecture 26: Holomorphically convex varieties

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Domains of holomorphy in $\mathbb{C}^n$

DEFINITION: Let $\Omega \subset \mathbb{C}^n$ be an open subset. It is called **a domain of holomorphy** if for any connected open subset $V \subset \mathbb{C}^n$ such that $W := \Omega \cap V$ is connected, there exists a function $f \in H^0(\mathcal{O}_W)$ which cannoe be extended to V.

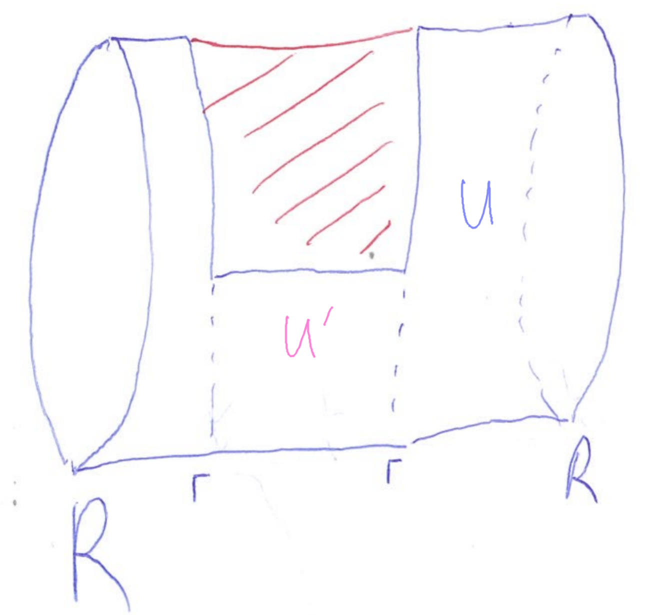
EXAMPLE: Every open subset $\Omega \subset \mathbb{C}$ is a domain of holomorphy. Indeed, for any $x \in \overline{\Omega} \backslash \Omega$, the function $z \mapsto (z-x)^{-1}$ cannot be extended to x.

EXAMPLE: Every convex subset $\Omega \subset \mathbb{C}^n$ is a domain of holomorphy. Indeed, for any $x \in \overline{\Omega} \backslash \Omega$, there exists a complex hyperplane $V \subset \mathbb{C}^n$ not intersecting Ω **(prove this)**. Suppose that V is defined by an affine equation $\lambda(z) = 0$. Then λ^{-1} is a holomorphic function on Ω which cannot be extended to x.

REMARK: The Hartogs figure, defined in the next slide, is **an example of a set which is not a domain of holomorphy,**

Hartogs figure

DEFINITION: Let $U \subset \mathbb{C}^{n-1}$ be a connected open set, and $U' \subsetneq U$ its open subset. Let $\Delta_R \subset \mathbb{C}$ be an open disk of radius R, and $\overline{\Delta}_R$ its closure. Define **the Hartogs figure** as $\Omega := \Delta_R \backslash \overline{\Delta}_r \times U \cup D(R) \times U' \subset \mathbb{C} \times \mathbb{C}^{n-1}$ and **the filled Hartogs figure** as $\tilde{\Omega} := \Delta_R \times U$, where $0 \leqslant r < R$.



CLAIM: Every holomorphic function $f \in \mathcal{O}_\Omega$ can be extended to $\mathcal{O}_{\tilde{\Omega}}$.

Proof: Let

$$\tilde{f}(z_1, z') := \frac{1}{2\pi\sqrt{-1}} \int_{|\zeta_1|=\rho} \frac{f(\zeta_1, z')}{\zeta_1 - z_1} d\zeta_1,$$

where $(z_1, z') \in \tilde{\Omega} \in \mathbb{C} \times U$ and $r < \rho < R$. By Cauchy formula, $\tilde{f} = f$ on $\Delta_R \times U'$, hence $\tilde{f} = f$ on Ω by analytic continuation. ■

Holomorphic convexity

DEFINITION: Let $K \subset X$ be a compact set in a complex variety. **A holomorphic hull** of K is

$$\widehat{K} := \left\{ z \in X \quad | \quad |f(z)| \leqslant \sup_{z \in K} |f(z)| \quad \forall f \in \mathcal{O}_X \right\}.$$

CLAIM: **$\widehat{K}$ is a closed subset of X containing K. Moreover, $\sup_{z \in K} |f(z)| = \sup_{z \in \widehat{K}} \quad \forall f \in \mathcal{O}_X$.**

Proof: $\widehat{K}$ is closed because $\widehat{K} = \bigcap_{f \in \mathcal{O}_X} |f|^{-1} \left(\left[0, \sup_{z \in K} |f(z)| \right] \right)$. ■

EXERCISE: Prove that **for any holomorphic map $h: X \longrightarrow Y$, we have $h(\widehat{K}) = \widehat{h(X)}$.**

DEFINITION: A subset $U \subset V$ is called **relatively compact** if its closure in V is compact; this relation is denoted $U \Subset V$.

CLAIM: Let $h: U \longrightarrow X$ be a holomorphic map, and $W \Subset U$ a relatively compact subset. Assume that $h(\partial W) \subset K$, where $\partial W = \overline{W} \backslash W$. **Then $h(W) \subset \widehat{K}$.**

Proof: Indeed, **any function $f \in \mathcal{O}_U$ satisfies $\sup_{z \in \overline{W}} = \sup_{z \in \partial W}$ by maximum principle.** ■

Holomorphic hull and convex hull

Claim 1: For any $K \subset \Omega \subset \mathbb{C}^n$, **the holomorphic hull $\hat{K}$ is contained in the convex hull $\hat{K}_{\text{aff}}$.**

Proof. Step 1: The set $\{x \in \mathbb{C}^n \mid f(x) \leqslant 1\}$ is a half-space if $f = e^\lambda$, where λ is an affine function. Indeed, this is the same as the set $\text{Re}(\lambda) \leqslant 0$.

Step 2: The set $\hat{K}_{\text{aff}}$ is intersection of all half-spaces containing K. By Step 1, this gives $\hat{K}_{\text{aff}} = \bigcap_{\lambda \in \text{Aff}} |e^\lambda|^{-1}\left(\left[0, \sup_{z \in K} |e^\lambda(z)|\right]\right)$.

Step 3: We obtain that

$$\hat{K} = \bigcap_{f \in \mathcal{O}_X} |f|^{-1}\left(\left[0, \sup_{z \in K} |f(z)|\right]\right) \subset \hat{K}_{\text{aff}} = \hat{K}_{\text{aff}} = \bigcap_{\lambda \in \text{Aff}} |e^\lambda|^{-1}\left(\left[0, \sup_{z \in K} |e^\lambda(z)|\right]\right) \blacksquare$$

Holomorphically convex varieties

DEFINITION: A variety X is called **holomorphically convex** if the holomorphic hull of a compact subset $K \subset X$ is always compact.

REMARK: In the following claim, **we assume that X has a countable base.**

CLAIM: A complex variety X **is holomorphically convex if and only if $X = \bigcup_i K_i^\circ$, where $K_0 \subset K_1 \subset ... \subset K_n \subset ...$ is a sequence of holomorphically convex compact subsets and K_i° is the interior of K_i.**

Proof. Step 1: If such a sequence exists, it gives an open covering $X = \bigcup_i K_i^\circ$. Then any compact set $K \subset X$ belongs to K_i° for some i, **hence $K \subset K_i$ which is compact and holomorphically convex.**

Conversely, if X is holomorphically convex, we can obtain X as a union of an increasing sequence $U_0 \subset U_1 \subset ...$ of open subsets with compact closure (here we use the countable base). **Then $X = \bigcup_i K_i^\circ$, where $K_i := \widehat{\overline{U}_i}$.** ■

Domains of holomorphy are holomorphically convex

Proposition 1: Let $\Omega \subset \mathbb{C}^n$ be a domain of holomorphy. **Then Ω is holomorphically convex.**

Proof. Step 1: Let $K \subset \Omega$ be a compact subset, and $R := d(K, \mathbb{C}^n \backslash \Omega)$ be the infimum of $d(x,y)$, where $x \in K$ and $y \in \mathbb{C}^n \backslash \Omega$. Clearly, $R > 0$. Denote by $B_1 \subset \mathbb{C}^n$ the unit ball. Fix $f \in \mathcal{O}_\Omega$, and let M be the maximum of $|f|$ on a compact set $K + r\overline{B}_1 \subset \Omega$

Consider the function $\varphi(t) := f(z + t\xi)$, where $z \in K$, $\xi \in B_1 \subset \mathbb{C}^n$ and $t \in \mathbb{C}$. Then φ is holomorphic in a disk of radius r, and bounded by M. Write $\varphi(t) = \sum a_k t^k$. **Cauchy formula implies $a_k \leqslant M r^{-k}$.**

Step 2: The function $z \mapsto a_k(z)$ is complex analytic, hence the inequality $a_k \leqslant M r^{-k}$ holds on $\widehat{K}$ for any $f \in \mathcal{O}_\Omega$. Then any analytic function on Ω can be extended to $\widehat{K} + rB_1$, which implies that $\widehat{K} + rB_1 \subset \Omega$ because Ω is a domain of holomorphy. **Then $\widehat{K} + rB_1 \subset \Omega$, for all $r < R$, which implies that $d(\widehat{K}, \mathbb{C}^n \backslash \Omega) = d(K, \mathbb{C}^n \backslash \Omega)$.**

Step 3: The set $\widehat{K}$ is bounded, because it belongs to a convex hull of K, which is compact. The distance between $\widehat{K}$ and $\mathbb{C}^n \backslash \Omega$ is positive and $\widehat{K}$ is closed in Ω, hence it is also closed in $\mathbb{C}^n$; a bounded and closed set is compact.

■

Subsets with accumulation points on a boundary

Claim 2: Let $U \subset \mathbb{R}^n$ be an open subset, and $\partial U := \overline{U}\backslash U$ its boundary. **Then U contains a countable subset $S := \{x_i\}$ which is discrete in U and satisfies $\partial S = \partial U$.**

Proof. Step 1: Using a countable base, we choose in ∂U a countable, dense subset $R = \{z_i\}$. **It would suffice to find a sequence $S := \{x_i\}$ which is discrete in U and satisfies $\partial S \supset R$.**

Step 2: We choose $\{x_i\}$ inductively. Suppose that $x_1, ..., x_{n-1}$ are already chosen, and let $a_n := \frac{1}{3} d(\{x_1, ..., x_{n-1}\}, \partial U)$. **This number is always positive, because ∂U is closed and $\{x_1, ..., x_{n-1}\}$ is compact.**

Step 3: Let $f: \mathbb{Z}^{>0} \longrightarrow \mathbb{Z}^{>0} \times \mathbb{Z}^{>0}$ be a bijective map, and $f(i) = (g_i, h_i)$. For each $n \in \mathbb{Z}^{>0}$, choose a point x_n such that $d(x_n, z_{h_n}) < a_n$ and $d(x_i, x_n) > a_n$, for all $i < n$. **The first property implies that**

$$\lim_{\textit{all } n, \textit{ such that } h_n = u} x_n = z_u,$$

and the second property implies that the a_n-neighbourhood of $\{x_1, ..., x_{n-1}\}$ does not contain other points of R. ∎

Properties of domains of holomorphy

THEOREM: Let $\Omega \subset \mathbb{C}^n$ be an open set.
Then the following are equivalent.
(a) Ω is a domain of holomorphy.
(b) Ω is holomorphically convex.
(c) For any countable set $\{z_i\} \subset \Omega$ without accumulation points in Ω, and any set of complex numbers $\{a_i\} \subset \mathbb{C}$ there exists a holomorphic function $F \in \mathcal{O}_\Omega$ wuch that $F(z_i) = a_i$.
(d) There exists $f \in \mathcal{O}_\Omega$ which is unbounded in a neighbourhood of any point of the boundary $\partial\Omega := \overline{\Omega} \backslash \Omega$.

Proof. Step 1: (d) $\Rightarrow$ (a) is clear. To prove (c) $\Rightarrow$ (d) we need to construct a set $\{z_i\}$ which has accumulation points in all $z \in \partial\Omega$ and only in them, which is done in Claim 2. (a) $\Rightarrow$ (b) is Proposition 1. It remains to show that (b) $\Rightarrow$ (c).

Properties of domains of holomorphy (2)

Let Ω is holomorphically convex. **It remains to show that for any countable set $\{z_i\} \subset \Omega$ without accumulation points in Ω, and any set of complex numbers $\{a_i\} \subset \mathbb{C}$ there exists a holomorphic function $F \in \mathcal{O}_\Omega$ wuch that $F(z_i) = a_i$.**

Step 2: Let $K_1 \subset K_2 \subset ...$ be a sequence of holomorphically convex compact sets, with $\Omega = \bigcup K_i$ (Claim 1). For each z_i there exists a unique a_i such that $z_i \in K_{a_i} \backslash K_{a_i - 1}$. Then there exists a holomorphic function $g_i \in \mathcal{O}_\Omega$ such that

$$\sup_{z \in K_{a_i - 1}} |g_i(z)| < |g_i(z_i)|.$$

Multiplying this function by a constant, we may assume that $g_i(z_i) = 1$. Consider an interpolation polynomial $P_i \in \mathbb{C}[z_1, ..., z_n]$ which is equal to 1 at z_i and 0 at $z_0, ..., z_{i-1}$. **Let $F := \sum_{j=0}^\infty \lambda_j P_j g_j^{m_j}$, where $\lambda_i \in \mathbb{C}$ and $m_j \in \mathbb{Z}^{>0}$ are chosen inductively in such a way that $\lambda_i = a_i - \sum_{j=0}^{i-1} \lambda_j P_j g_j^{m_j}$ and $|\lambda_i P_i g_i^{m_i}| \leqslant 2^{-i}$ on $K_{a_i - 1}$.** Once we have found λ_i, the second condition holds for m_i sufficiently big, because $|g_i|\Big|_{K_{a_i - 1}} < 1 - \varepsilon$.

Step 3: Since $\{z_i\}$ has no accumulation points, the sequence a_i tends to ∞. Then the sum $\sum_{j=0}^\infty \lambda_j P_j g_j^{m_j}$ uniformly converges on compact sets. **A uniform limit of holomorphic functions is holomorphic by Cauchy formula.** ■