Complex analytic spaces

lecture 27: Maximum principle

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November 13, 2023, 13:30

Algebra of differential operators

DEFINITION: Let M be a smooth manifold. **The algebra of differential operators** is the subalgebra in $\operatorname{End}(C^\infty M)$ generated by vector fields and multiplication by a function. **A differential operator of order d** is an operator which is locally generated by a product of no more than d vector fields.

REMARK: Denote by $\operatorname{Diff}^i(M)$ the space of differential operators of order i. **Clearly,** $\operatorname{Diff}^i(M)\operatorname{Diff}^j(M) = \operatorname{Diff}^{i+j}(M)$**.**

EXERCISE: Let $F \in \operatorname{Diff}^i(M)$ and $G \in \operatorname{Diff}^j(M)$. **Prove that** $[F,G] \in \operatorname{Diff}^{i+j-1}(M)$**.**

Filtered algebras

DEFINITION: An (increasing) filtration on a vector space V is a sequence of subspaces $V_0 \subset V_1 \subset V_2 \subset ...$ such that $\bigcup V_i = V$. **A filtered algebra** is an algebra A with a filtration $A_0 \subset A_1 \subset A_2 \subset ...$ such that $A_i \cdot A_j \subset A_{i+j}$.

EXAMPLE: The algebra of differential operators is filtered, $\operatorname{Diff}^0(M) \subset \operatorname{Diff}^1(M) \subset \operatorname{Diff}^2(M) \subset ...$

DEFINITION: Let $A = \bigcup_i A_i$ be an associative filtered algebra. The **associated graded space** $\bigoplus_i A_i/A_{i-1}$ is equipped with a multiplicative structure: a product of a mod A_{i-1} and b mod A_{j-1} gives ab mod A_{i+j-1}. The algebra $\bigoplus_i A_i/A_{i-1}$ is called **the associated graded algebra** of the filtered algebra A.

EXERCISE: Prove that **the associated graded algebra $\bigoplus_i \operatorname{Diff}^i(M)/\operatorname{Diff}^{i-1}(M)$ of the algebra of differential operators is commutative.**

DEFINITION: This algebra is called **the algebra of symbols of differential operators**.

The algebra of symbols

THEOREM: The bundle $\operatorname{Diff}^i(M)/\operatorname{Diff}^{i-1}(M)$ is isomorphic to $\operatorname{Sym}^i TM$.

Proof: For $i=1$ this is clear from the definition: first order differential operators are obtained as a sum of derivations and operators of multiplication by a function; this gives an exact sequence $0 \longrightarrow C^\infty M \longrightarrow \operatorname{Diff}^1 M \longrightarrow TM \longrightarrow 0$. For $i>0$, we notice that the multiplication map

$$\operatorname{Sym}^i TM = \operatorname{Sym}^i(\operatorname{Diff}^1(M)/\operatorname{Diff}^0(M)) \longrightarrow \operatorname{Diff}^i(M)/\operatorname{Diff}^{i-1}(M)$$

is by construction surjective; it is injective, because it is injective on polynomial functions. ■

EXERCISE: Prove that **the ring $\oplus_i \operatorname{Diff}^i(M)/\operatorname{Diff}^{i-1}(M)$ of symbols is isomorphic to the ring of functions on the total space of T^*M which are polynomial on fibers.**

DEFINITION: Let $D \in \operatorname{Diff}^i(M)$ be a differential operator of order i. Its **symbol** is its image in $\operatorname{Diff}^i(M)/\operatorname{Diff}^{i-1}(M) = \operatorname{Sym}^i TM$.

DEFINITION: Let $D \in \operatorname{Diff}^i(M)$ be a differential operator, and $\sigma \in \operatorname{Sym}^i TM$ its symbol. The operator D is called **elliptic** if $\sigma(x, x..., x) \neq 0$ for any $x \in T^*_m M \backslash 0$.

EXERCISE: Let $D \in \operatorname{Diff}^i(M)$ be an elliptic operator. **Prove that i is even.**

Elliptic operators of second order

EXAMPLE: A second order differential operator on $C^\infty\mathbb{R}^n$ is written as

$$D(f) = af + \sum_i b_i \frac{\partial f}{\partial x_i} + \sum_{i,j} c_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j},$$

where a, b_i, $c_{i,j}$ are smooth functions, and $x_i, i = 1, ..., n$ coordinates. Its symbol is given by a quadratic form $\xi \longrightarrow \sum_{i,j} c_{i,j}\xi_i\xi_j$, where $\xi = (\xi_1, ..., \xi_n) \in T_x^*\mathbb{R}^n$. **This operator is elliptic if the matrix $(c_{i,j})$ is positive or negative definite in any point of $\mathbb{R}^n$.**

REMARK: By convention, we assume that $(c_{i,j})$ **is always positive definite when D is elliptic**.

REMARK: For any elliptic operator D, its symbol $\sigma(D) \in \mathrm{Sym}^2(TM)$ **defines a Riemannian metric on M.**

Strong maximum principle

THEOREM:
(strong maximum principle for second order elliptic equations; Eberhard Hopf, 1927) Let M be a manifold, not necessarily compact, and $D: C^\infty M \longrightarrow C^\infty M$ an elliptic operator of second order, which satisfies $D(\text{const}) = 0$. Consider a function $u \in C^\infty M$ such that $D(u) \geqslant 0$. **Assume that u has a local maximum somewhere on M. Then u is a constant.**

Maximum principle will be proven at the end of today's lecture. We start with a special case.

Proof of maximum principle, for the case $D(u) > 0$: In coordinates, D is written as $Du = \sum_{i,j} A^{ij} u_{ij} + \sum_i B^i u_i,$, where u is the matrix of second derivatives of u, $u_i = \frac{\partial u}{\partial x_i}$, and A^{ij} a function taking values in positive definite matrices. Let z be the point where u reaches a relative maximum. In this point the first derivatives of u vanish, and the matrix of second derivatives is negative, hence $Du|_z = \sum_{i,j} A^{ij} u_{ij}|_z \leqslant 0$, contradicting $Du > 0$. ■

I will first prove **the weak maximum principle,** and then deduce the strong maximum principlle.

Weak maximum principle

THEOREM: (The weak maximum principle)
Let $D: C^\infty\mathbb{R}^n \longrightarrow C^\infty\mathbb{R}^n$ be an elliptic operator of second order, which satisfies $D(\text{const}) = 0$. Consider a relatively compact open subset $\Omega \Subset \mathbb{R}^n$. **Then any solution u of the inequality $D(u) \geqslant 0$ reaches its maximum $\sup_\Omega u$ on the boundary $\partial\Omega$.**

Proof. Step 1: Let $z \in \overline{\Omega}$ be a point where u reaches maximum, and x_i coordinates in its neighbourhood U, with origin in z. Rescaling the coordinates if necessary, we can always assume that Ω is relatively compact in the unit ball B_1. **It would suffice to show that $u(z) = u(v)$ for some $z \in \partial\Omega$.**

Step 2: Adding to u a solution φ of the inequality $D\varphi > 0$, **we obtain a function $u+\varphi$ which reaches its maximum on ∂U,** because of the strong maximum principle for solutions of $Du > 0$.

Step 3: The function $\varphi := \varepsilon e^{cx_1}$ satisfies $D\varphi > 0$ if c is chosen such that $A^{1,1}c > |B^1|$. Indeed, $\varepsilon^{-1}D(\varphi) = c^2A^{1,1}e^{cx_1} + cB^1e^{cx_1} > 0$.

Step 4: Since the maximum of $u + \varepsilon e^{cx_1}$ is reached on $\partial\Omega$ for any $\varepsilon > 0$. we obtain that $\sup_\Omega u = \sup_{\partial\Omega} u$. ■

Hopf lemma

LEMMA: (Hopf lemma)
Let $D(u) = \sum_{i,j} A^{ij} u_{ij} + \sum_i B^i u_i$ be an elliptic operator on a unit ball $B \subset \mathbb{R}^n$, and $u \in C^\infty B$ a function which satisfies $D(u) \geqslant 0$. Assume that u reaches maximum $u(z_0) = 0$ in $z_0 \in \partial B$, and inside B we have $u < 0$. Denote by $\vec{r}$ the radial vector field, $\vec{r} = \sum x_i \frac{d}{dx_i}$. **Then the derivative $D_{\vec{r}} u\big|_{z_0}$ in the radial direction is positive.**

Proof. Step 1: Consider a non-negative function $v \in C^\infty B$, defined by $v(x) = e^{-\alpha|x|^2} - e^{-\alpha}$, where $\alpha > 0$ is a real number. Then

$$D(v)|_x = \alpha^2 e^{-\alpha r(x)^2} \sum A^{ij} x_i x_j + e^{-\alpha r(x)^2} (\alpha\zeta + \xi),$$

where $\zeta, \xi \in \mathbb{C}^\infty B$ are bounded functions on B, independent from α. Therefore **for sufficiently big $\alpha > 0$, we have $D(v) > 0$ in the set $\Omega = B \backslash B'$,** where $B' \subset B$ is an open ball with center in 0 and radius $r_0 < 1$.

Step 2: For a sufficiently small $\varepsilon > 0$, we have $u + \varepsilon v < 0$ in B', because $u < \delta < 0$ on B'. Since $v = 0$ on ∂B, weak maximum principle implies that $u + \varepsilon v < 0$ in Ω, and $u + \varepsilon v$ reaches its maximum in z_0. **This implies that $D_{\vec{r}}(u + \varepsilon v)\big|_{z_0} \geqslant 0$.**

Step 3: An easy computation gives $D_{\vec{r}} v\big|_{z_0} < 0$, hence Step 2 implies $D_{\vec{r}} u\big|_{z_0} > 0$.
■

Strong maximum principle (proof)

THEOREM:
(strong maximum principle for second order elliptic equations; Eberhard Hopf, 1927) Let M be a manifold, not necessarily compact, and $D: C^\infty M \longrightarrow C^\infty M$ an elliptic operator of second order, which satisfies $D(\text{const}) = 0$. Consider a function $u \in C^\infty M$ such that $D(u) \geqslant 0$. **Assume that u has a local maximum somewhere on M. Then u is a constant.**

Proof. Step 1: Suppose that the local maximum is reached in $z \in M$, and $Z := \{m \in M \quad | \quad u(m) = u(z)\}$. If $u \neq \text{const}$, **there exists an open ball $B \subset M$ with interior not intersecting Z, and boundary intersecting Z.** Choose this ball in such a way that $u < u(z)$ inside B.

Step 2: Since the derivative of z in z_0 is non-zero (by Hopf lemma), this point cannot be a local maximum of u. ∎