Complex analytic spaces

lecture 28: Pluri-Laplacian

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Hodge decomposition (reminder)

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: Let $\Lambda^{p,0}(M,I) := \Lambda^p_{C^\infty_{\mathbb{C}}(M)}(T^{1,0}M)^*$, $\Lambda^{0,p}(M,I) := \Lambda^p_{C^\infty_{\mathbb{C}}(M)}(T^{0,1}M)^*$, and $\Lambda^{p,q}(M,I) := \Lambda^{p,0}(M,I) \otimes_{C^\infty_{\mathbb{C}}(M)} \Lambda^{0,q}(M,I)$.

CLAIM:

$$\Lambda^n M \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=n} \Lambda^{p,q}(M,I)$$

Complex manifolds (reminder)

EXAMPLE: Let $M = \mathbb{C}^n$, with the complex coordinates $z_1, ..., z_n$ and real coordinates $x_i := \operatorname{Re}(z_i), y_i := \operatorname{Im}(z_i)$. **The standard almost complex structure** is defined as $I(dx_i) = dy_i,\ I(dy_i) = dx_i,.$

DEFINITION: A complex manifold is an almost complex manifold which is locally isomorphic to $\mathbb{C}^n$ with this complex structure.

REMARK: A 1-form $\alpha \in \Lambda^1(M, \mathbb{C})$ satisfies $\alpha(Ix) = \sqrt{-1}\,\alpha(x)$ if and only if $\alpha \in \Lambda^{1,0}(M)$. Therefore, **a function $f: M \longrightarrow \mathbb{C}$ is complex differentiable if and only if $df \in \Lambda^{1,0}(M)$.**

Graded vector spaces and algebras

DEFINITION: A **graded vector space** is a space $V^* = \bigoplus_{i\in\mathbb{Z}} V^i$.

REMARK: If V^* is graded, the endomorphisms space $\operatorname{End}(V^*) = \bigoplus_{i\in\mathbb{Z}} \operatorname{End}^i(V^*)$ is also graded, with $\operatorname{End}^i(V^*) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}(V^j, V^{i+j})$

DEFINITION: A **graded algebra** (or "graded associative algebra") is an associative algebra $A^* = \bigoplus_{i\in\mathbb{Z}} A^i$, with the product compatible with the grading: $A^i \cdot A^j \subset A^{i+j}$.

REMARK: A bilinear map of graded paces which satisfies $A^i \cdot A^j \subset A^{i+j}$ is called **graded**, or **compatible with grading**.

REMARK: The category of graded spaces can be defined as a **category of vector spaces with $U(1)$-action,** with the weight decomposition providing the grading. Then **a graded algebra is an associative algebra in the category of spaces with $U(1)$-action**.

DEFINITION: An operator on a graded vector space is called **even** (**odd**) if it shifts the grading by even (odd) number. The **parity** $\tilde{a}$ of an operator a is 0 if it is even, 1 if it is odd. We say that an operator is **pure** if it is even or odd.

Supercommutator

DEFINITION: A **supercommutator** of pure operators on a graded vector space is defined by a formula $\{a,b\} = ab - (-1)^{\tilde{a}\tilde{b}} ba$.

DEFINITION: A graded associative algebra is called **graded commutative** (or "supercommutative") if its supercommutator vanishes.

EXAMPLE: The Grassmann algebra is supercommutative.

DEFINITION: A graded Lie algebra (Lie superalgebra) is a graded vector space $\mathfrak{g}^*$ equipped with a bilinear graded map $\{\cdot,\cdot\}: \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ which is graded anticommutative: $\{a,b\} = -(-1)^{\tilde{a}\tilde{b}}\{b,a\}$ and satisfies **the super Jacobi identity** $\{c,\{a,b\}\} = \{\{c,a\},b\} + (-1)^{\tilde{a}\tilde{c}}\{a,\{c,b\}\}$

EXAMPLE: Consider the algebra $\operatorname{End}(A^*)$ of operators on a graded vector space, with supercommutator as above. **Then $\operatorname{End}(A^*), \{\cdot,\cdot\}$ is a graded Lie algebra.**

Lemma 1: Let d be an odd element of a Lie superalgebra, satisfying $\{d,d\} = 0$, and L an even or odd element. **Then $\{\{L,d\},d\} = 0$.**

Proof: $0 = \{L,\{d,d\}\} = \{\{L,d\},d\} + (-1)^{\tilde{L}}\{d,\{L,d\}\} = 2\{\{L,d\},d\}$. ∎

The twisted differential d^c

DEFINITION: The **twisted differential** is defined as $d^c := IdI^{-1}$.

CLAIM: Let (M,I) be a complex manifold. **Then $\partial := \frac{d+\sqrt{-1}\,d^c}{2}$, $\overline{\partial} := \frac{d-\sqrt{-1}\,d^c}{2}$ are the Hodge components of d**, $\partial = d^{1,0}$, $\overline{\partial} = d^{0,1}$.

Proof: The Hodge components of d are expressed as $d^{1,0} = \frac{d+\sqrt{-1}\,d^c}{2}$, $d^{0,1} = \frac{d-\sqrt{-1}\,d^c}{2}$. Indeed, $I(\frac{d+\sqrt{-1}\,d^c}{2})I^{-1} = \sqrt{-1}\,\frac{d+\sqrt{-1}\,d^c}{2}$, hence $\frac{d+\sqrt{-1}\,d^c}{2}$ **has Hodge type (1,0);** the same argument works for $\overline{\partial}$. ■

CLAIM: Let W be **the Weil operator**, $W\big|_{\Lambda^{p,q}(M)} = \sqrt{-1}\,(p-q)$. On any complex manifold, one has $d^c = [W,d]$.

Proof: Clearly, $[W,d^{1,0}] = \sqrt{-1}\,d^{1,0}$ and $[W,d^{0,1}] = -\sqrt{-1}\,d^{0,1}$. Then $[W,d] = \sqrt{-1}\,d^{1,0} - \sqrt{-1}\,d^{0,1} = IdI^{-1}$. ■

COROLLARY: $\{d,d^c\} = \{d,\{d,W\}\} = 0$ (Lemma 1).

Plurilaplacian

THEOREM: Let (M,I) be a complex manifold. **Then 1.** $\partial^2 = 0$.
2. $\overline{\partial}^2 = 0$.
3. $dd^c = -d^c d$
4. $dd^c = 2\sqrt{-1}\,\partial\overline{\partial}$.

Proof: The first is vanishing of (2,0)-part of d^2, and the second is vanishing of its (0,2)-part. Now, $\{d, d^c\} = -\{d, \{d, W\}\} = 0$ (Lemma 1), this gives $dd^c = -d^c d$. Finally, $2\sqrt{-1}\,\partial\overline{\partial} = \frac{1}{2}(d + \sqrt{-1}\,d^c)(d - \sqrt{-1}\,d^c) = \frac{1}{2}(dd^c - d^c d) = dd^c$. ■

DEFINITION: The operator dd^c is called **the pluri-Laplacian**.

REMARK: The pluri-Laplacian **takes real functions to real (1,1)-forms on M.**

EXERCISE: Prove that **on a Riemannian surface (M, I, ω), one has $dd^c(f) = \Delta(f)\omega$.**

DEFINITION: **The Hodge $U(1)$-action** on differential forms on a complex manifold defined by $\rho(t)(\eta) = e^{tW}(\eta)$. On (p,q)-forms, it acts as a scalar $\rho(t)\Big|_{\Lambda^{p,q}(M)} = e^{(p-q)\sqrt{-1}}\,\mathrm{Id}$; the (p,p)-forms are clearly invariant.

Positive (1,1)-forms

CLAIM: Consider a real (1,1)-form $\eta \in \Lambda^{1,1}(M)\cap\Lambda^2(M,\mathbb{R})$. **Then the bilinear form $g_\eta(x,y) := \eta(x,Iy)$ is symmetric.**
Proof: Clearly, $0 = W(\eta)(x,y) = \eta(W(x),y) + \eta(x,W(y)) = \eta(Ix,y) + \eta(x,Iy)$. This gives $\eta(x,Iy) = -\eta(Ix,y) = \eta(y,Ix)$. ■

CLAIM: This construction **defines a bijection between $U(1)$-invariant symmetric forms $g \in \operatorname{Sym}^2(T^*M)$ and sections of $\Lambda^{1,1}(M) \cap \Lambda^2(M,\mathbb{R})$.** ■

DEFINITION: A real (1,1)-form η is called **positive** if $\eta(x,Ix) \geqslant 0$ for any $x \in TM$.

REMARK: By convention, **0 is a positive (1,1)-form.**

DEFINITION: A (1,1)-form is called **Hermitian** if it is positive and non-degenerate, that is, when $\eta(x,Ix) > 0$ for any $x \in TM\backslash 0$.

REMARK: The above construction **gives a bijective correspondence between the Hermitian (1,1)-forms and $U(1)$-invariant Riemannian metric tensors on M.**

EXAMPLE: For any (1,0)-form ξ, **the form $\sqrt{-1}\,\xi\wedge\bar\xi$ is positive (prove this).**

The coordinate operators

Let V be an even-dimensional real vector space equipped with a scalar product, and $v_1, ..., v_{2n}$ an orthonormal basis. Denote by $e_{v_i}: \Lambda^k V \longrightarrow \Lambda^{k+1} V$ an operator of multiplication, $e_{v_i}(\eta) = v_i \wedge \eta$. Let $i_{v_i}: \Lambda^k V \longrightarrow \Lambda^{k-1} V$ be an adjoint operator, $i_{v_i} = * e_{v_i} *$.

CLAIM: The operators e_{v_i}, i_{v_i}, Id are a basis of an **odd Heisenberg Lie superalgebra $\mathfrak{H}$**, with **the only non-trivial supercommutator given by the formula $\{e_{v_i}, i_{v_j}\} = \delta_{i,j} \mathrm{Id}$.**

Now, consider the tensor $\omega = \sum_{i=1}^n v_{2i-1} \wedge v_{2i}$, and let $L(\alpha) = \omega \wedge \alpha$, and $\Lambda := L^*$ be the corresponding **Hodge operators**.

CLAIM: (Lefschetz $\mathfrak{sl}(2)$-action)
From the commutator relations in $\mathfrak{H}$, one obtains immediately that

$$H := [L, \Lambda] = \left[\sum e_{v_{2i-1}} e_{v_{2i}}, \sum i_{v_{2i-1}} i_{v_{2i}}\right] = \sum_{i=1}^{2n} e_{v_i} i_{v_i} - \sum_{i=1}^{2n} i_{v_i} e_{v_i},$$

is a scalar operator acting as $k-n$ on k-forms.

COROLLARY: The triple L, Λ, H satisfies the relations for the $\mathfrak{sl}(2)$ Lie algebra: $[L, \Lambda] = H$, $[H, L] = 2L$, $[H, \Lambda] = 2\Lambda$.

Laplacian and a pluri-Laplacian

DEFINITION: Let ω be a Hermitian form on a complex manifold (M, I), and $\Lambda: \Lambda^{1,1}(M) \longrightarrow C^\infty M$ the Lefschetz operator. **The Laplacian** $\Delta: C^\infty M \longrightarrow C^\infty M$ is defined as $\Delta(f) := \Lambda(dd^c f)$.

REMARK: Consider an orthonormal frame $\xi_1, ..., \xi_n \in \Lambda^{1,0}M$; then $\omega = \sqrt{-1}\sum \xi_i \wedge \overline{\xi}_i$. Then $dd^c = 2\sqrt{-1}\,\partial\overline{\partial}$ has the same symbol as

$$f \mapsto \sum_{i,j} \frac{2}{\sqrt{-1}} \operatorname{Lie}_{x_i} \operatorname{Lie}_{\overline{x}_j}(f) \xi_i \wedge \overline{\xi}_j$$

where $x_1, ..., x_n \in T^{1,0}M$ is the dual basis. **This implies that $\Delta(f)$ has the same symbol as**

$$f \mapsto \sum_{i} \frac{2}{\sqrt{-1}} \operatorname{Lie}_{x_i} \operatorname{Lie}_{\overline{x}_i}(f)$$

which **has the same symbol as $\sum_i \operatorname{Lie}^2_{p_i}(f) + \operatorname{Lie}^2_{q_i}(f)$,** where $p_i = \operatorname{Re} x_i$, $q_i = \operatorname{Im}(x_i)$.

COROLLARY: The Laplacian $\Delta(f) = \Lambda(dd^c f)$ **is an elliptic operator of second order.** ■

Pluri-harmonic functions

DEFINITION: A function f on a complex manifold is called **pluri-harmonic** if $dd^2 f = 0$.

REMARK: A function f is called **holomorphic** if $\overline{\partial} f = 0$, and **antiholomorphic** if $\partial f = 0$. Since $dd^c = 2\sqrt{-1}\,\partial\overline{\partial} = -2\sqrt{-1}\,\overline{\partial}\partial$, **any holomorphic and any antiholomorphic function is pluri-harmonic.**

THEOREM: Any pluriharmonic function **is locally expressed as a sum of holomorphic and antiholomorphic function.**

Proof: Let f be a pluriharmonic function on a ball, and $\alpha = \partial f$ Since $\overline{\partial}(\alpha) = 0$, this form is holomorphic; since $\partial^2 = 0$, it is also closed. Poincaré lemma applied to holomorphic functions implies that $\alpha = du$, where u is holomorphic. Then $d(f - u)$ is a (0,1)-form, hence $v := f - u$ is antiholomorphic. **We obtain that $f = u + v$, where u is holomorphic, and v is antiholomorphic.** ■

In our proof, we use the following version of Poincaré lemma.
LEMMA: Let $B \subset \mathbb{C}^n$ be an open ball, and η a closed holomorphic $(d,0)$-form **Then $\eta = d\alpha$, where α is a holomorphic $(d-1,0)$-form.**

Proof: It follows from the same argument as one which proves the Poincaré lemma (Lecture 29). ■