Complex analytic spaces

lecture 29: Plurisubharmonic functions and pseudoconvexity

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Hodge decomposition (reminder)

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I : TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: Let $\Lambda^{p,0}(M,I) := \Lambda^p_{C^\infty_{\mathbb{C}}(M)}(T^{1,0}M)^*$, $\Lambda^{0,p}(M,I) := \Lambda^p_{C^\infty_{\mathbb{C}}(M)}(T^{0,1}M)^*$, and $\Lambda^{p,q}(M,I) := \Lambda^{p,0}(M,I) \otimes_{C^\infty_{\mathbb{C}}(M)} \Lambda^{0,q}(M,I)$.

CLAIM:

$$\Lambda^n M \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=n} \Lambda^{p,q}(M,I)$$

Plurilaplacian(reminder)

DEFINITION: The **twisted differential** is defined as $d^c := IdI^{-1}$.

CLAIM: Let (M, I) be a complex manifold. **Then** $\partial := \frac{d+\sqrt{-1}\, d^c}{2}$**,** $\overline{\partial} := \frac{d-\sqrt{-1}\, d^c}{2}$ **are the Hodge components of** d, $\partial = d^{1,0}$, $\overline{\partial} = d^{0,1}$.

CLAIM: Let W be **the Weil operator**, $W\Big|_{\Lambda^{p,q}(M)} = \sqrt{-1}\,(p-q)$. On any complex manifold, one has $d^c = [W, d]$**.**

THEOREM: Let (M, I) be a complex manifold. **Then 1.** $\partial^2 = 0$**.**
2. $\overline{\partial}^2 = 0$**.**
3. $dd^c = -d^c d$
4. $dd^c = 2\sqrt{-1}\,\partial\overline{\partial}$**.**

DEFINITION: The operator dd^c is called **the pluri-Laplacian**.

REMARK: The pluri-Laplacian **takes real functions to real (1,1)-forms on** M**.**

EXERCISE: Prove that **on a Riemannian surface** (M, I, ω)**, one has** $dd^c(f) = \Delta(f)\omega$**.**

Positive (1,1)-forms (reminder)

CLAIM: Consider a real (1,1)-form $\eta\in\Lambda^{1,1}(M)\cap\Lambda^2(M,\mathbb{R})$. **Then the bilinear form $g_\eta(x,y):=\eta(x,Iy)$ is symmetric.**
Proof: Clearly, $0=W(\eta)(x,y)=\eta(W(x),y)+\eta(x,W(y))=\eta(Ix,y)+\eta(x,Iy)$. This gives $\eta(x,Iy)=-\eta(Ix,y)=\eta(y,Ix)$. ■

CLAIM: This construction **defines a bijection between $U(1)$-invariant symmetric forms $g\in\operatorname{Sym}^2(T^*M)$ and sections of $\Lambda^{1,1}(M)\cap\Lambda^2(M,\mathbb{R})$.** ■

DEFINITION: A real (1,1)-form η is called **positive** if $\eta(x,Ix)\geqslant 0\,\forall x\in TM$.
REMARK: By convention, **0 is a positive (1,1)-form.**
DEFINITION: A (1,1)-form is called **Hermitian** if it is positive and non-degenerate, that is, when $\eta(x,Ix)>0$ for any $x\in TM\backslash 0$.

REMARK: The above construction **gives a bijective correspondence between the Hermitian (1,1)-forms and $U(1)$-invariant Riemannian metric tensors on M.**

EXAMPLE: For any (1,0)-form ξ, **the form $\sqrt{-1}\,\xi\wedge\overline{\xi}$ is positive**.

Now, consider the operator $L(\alpha):=\omega\wedge\alpha$, and let $\Lambda:=L^*$.
CLAIM: The Laplacian $\Delta(f)=\Lambda(dd^cf)$ **is an elliptic operator of second order.** ■

Pluri-harmonic functions (reminder)

DEFINITION: A function f on a complex manifold is called **pluri-harmonic** if $dd^2 f = 0$.

REMARK: A function f is called **holomorphic** if $\overline{\partial} f = 0$, and **antiholomorphic** if $\partial f = 0$. Since $dd^c = 2\sqrt{-1}\,\partial\overline{\partial} = -2\sqrt{-1}\,\overline{\partial}\partial$, **any holomorphic and any antiholomorphic function is pluri-harmonic.**

THEOREM: Any pluriharmonic function **is locally expressed as a sum of holomorphic and antiholomorphic function.**

Proof: Let f be a pluriharmonic function on a ball, and $\alpha = \partial f$ Since $\overline{\partial}(\alpha) = 0$, this form is holomorphic; since $\partial^2 = 0$, it is also closed. Poincaré lemma applied to holomorphic functions implies that $\alpha = du$, where u is holomorphic. Then $d(f - u)$ is a (0,1)-form, hence $v := f - u$ is antiholomorphic. **We obtain that $f = u + v$, where u is holomorphic, and v is antiholomorphic.** ∎

In our proof, we use the following version of Poincaré lemma.

LEMMA: Let $B \subset \mathbb{C}^n$ be an open ball, and η a closed holomorphic $(d, 0)$-form **Then $\eta = d\alpha$, where α is a holomorphic $(d-1, 0)$-form.**

Proof: It follows from the same argument as one which proves the Poincaré lemma (next slide). ∎

Poincaré lemma

DEFINITION: An open subset $U \subset \mathbb{R}^n$ is called **starlike** if for any $x \in U$ the interval $[0, x]$ belongs to U.

THEOREM: (Poicaré lemma) Let $U \subset \mathbb{R}^n$ be a starlike subset. **Then $H^i(U) = 0$ for $i > 0$.** In other words, **Tany closed i-form on U is exact.**

REMARK: The proof would follow if we construct a vector field $\vec{r}$ such that $\operatorname{Lie}_{\vec{r}}$ is invertible on $\Lambda^*(M)$: $\operatorname{Lie}_{\vec{r}} R = \operatorname{Id}$. Indeed, for any closed form α we would have $\alpha = \operatorname{Lie}_{\vec{r}} R\alpha = di_{\vec{r}} R\alpha + i_{\vec{r}} R d\alpha = di_{\vec{r}} R\alpha$, hence any closed form is exact.

Then Poincaré lemma is implied by the following statement.

PROPOSITION: Let $U \subset \mathbb{R}^n$ be a starlike subset, $t_1, ..., t_n$ coordinate functions, and $\vec{r} := \sum t_i \frac{d}{dt_i}$ the radial vector field. **Then $\operatorname{Lie}_{\vec{r}}$ is invertible on $\Lambda^i(U)$ for $i > 0$.**

Radial vector field on starlike sets

PROPOSITION: Let $U \subset \mathbb{R}^n$ be a starlike subset, $t_1, ..., t_n$ coordinate functions, and $\vec{r} := \sum t_i \frac{d}{dt_i}$ the radial vector field. **Then $\operatorname{Lie}_{\vec{r}}$ is invertible on $\Lambda^i(U)$ for $i > 0$.**

Proof. Step 1: Let t be the coordinate function on a real line, $f(t) \in C^\infty\mathbb{R}$ a smooth function, and $v := t\frac{d}{dt}$ a vector field. Define $R(f)(t) := \int_0^1 \frac{f(\lambda t)}{\lambda} d\lambda$. Then this integral converges whenever $f(0) = 0$, and satisfies $\operatorname{Lie}_v R(f) = f$. Indeed,

$$\int_0^1 \frac{f(\lambda t)}{\lambda} d\lambda = \int_0^t \frac{f(\lambda t)}{t\lambda} d(t\lambda) = \int_0^t \frac{f(z)}{z} dz,$$

hence $\operatorname{Lie}_v R(f) = t\frac{f(t)}{t} = f(t)$.

Step 2: Consider a function $f \in C^\infty\mathbb{R}^n$ satisfying $f(0) = 0$, and $x = (x_1, ..., x_n) \in \mathbb{R}^n$. **Then**

$$R(f)(x) := \int_0^1 \frac{f(\lambda x)}{\lambda} d\lambda$$

converges, and satisfies $\operatorname{Lie}_{\vec{r}} R(f) = f$.

Radial vector field on starlike sets (2)

Step 3: Consider a differential form $\alpha \in \Lambda^i$, and let $h_\lambda x \longrightarrow \lambda x$ be the homothety with coefficient $\lambda \in [0,1]$. Define

$$R(\alpha) := \int_0^1 \lambda^{-1} h_\lambda^*(\alpha) d\lambda.$$

Since $h_\lambda^*(\alpha) = 0$ for $\lambda = 0$, this integral converges. **It remains to prove that $\operatorname{Lie}_{\vec{r}} R = \operatorname{Id}$.**

Step 4: Let α be a coordinate monomial, $\alpha = dt_{i_1} \wedge dt_{i_2} \wedge ... \wedge dt_{i_k}$. Clearly, $\operatorname{Lie}_{\vec{r}}(T^{-1}\alpha) = 0$, where $T = t_{i_1} t_{i_2} ... t_{i_k}$. **Since $h_\lambda^*(f\alpha) = h_\lambda^*(Tf)T^{-1}\alpha$, we have $R(f\alpha) = R(Tf)T^{-1}\alpha$ for any function $f \in C^\infty M$.** This gives

$$\operatorname{Lie}_{\vec{r}} R(f\alpha) = \operatorname{Lie}_{\vec{r}} R(Tf)T^{-1}\alpha = TfT^{-1}\alpha = f\alpha.$$

■

Plurisubharmonic functions

DEFINITION: A function f on a complex manifold is called **plurisubharmonic** (or **psh**) if $dd^c f$ is a positive (1,1)-form, and **strictly plurisubharmonic** if $dd^c f$ is a positive definite (and ipso facto Kähler) form.

REMARK: For any plurisubharmonic function f, and any Hermitian form ω, we have $\Delta(f) \geqslant 0$, where Δ is an elliptic operator. Applying the strong maximum principle, we obtain
COROLLARY: A plurisubharmonic function on a manifold **cannot have a local maximum, unless it is constant.** ■

EXAMPLE: A sum of plurisubharmonic functions is plurisubharmonic.
EXAMPLE: Let f be a holomorphic function. Then $dd^c|f|^2 = 2\sqrt{-1}\,\partial\bar\partial f\bar f = 2\sqrt{-1}\,(\partial f \wedge \overline{\partial f})$, hence $|f|^2$ is plurisubharmonic.

COROLLARY: Let $f_1, ..., f_n$ be a collection of holomorphic functions on a complex manifold. **Then $\sum_i |f_i|^2$ is plurisubharmonic, hence it cannot have a maximum.**
EXAMPLE: Let $\mu \in C^\infty\mathbb{R}$. Then

$$dd^c(\mu(f))|_m = \mu'(f(z))^2 dd^c f + \mu''(f(z)) df \wedge d^c f.$$

Therefore, **for any psh function f, the composition $\mu(f)$ is psh when $\mu'' \geqslant 0$ and $\mu' > 0$.**

Kähler potential

I will use the following difficult theorem without a proof.

LEMMA: ("Poincaré-Dolbeault-Grothendieck lemma")
Let η be a $\bar\partial$-closed (p,q)-form, $q>0$, on an open ball $B\subset\mathbb{C}^n$. **Then $\eta\in\operatorname{im}\bar\partial$.**

DEFINITION: A closed Hermitian (1,1)-form ω is called **a Kähler form**. A function f is called **its Kähler potential** when $\omega=dd^c f$.

CLAIM: Any Kähler form on an open ball $B\subset\mathbb{C}^n$ admits a Kähler potential.

Proof. Step 1: Poincaré-Dolbeault-Grothendieck lemma implies that $\omega=\bar\partial\eta$, for some $\eta\in\Lambda^{1,0}B$. Then $\partial\bar\partial\eta=\partial\omega=0$, which implies $\bar\partial\partial\eta=0$.

Step 2: We obtain that $\partial\eta$ is a holomorphic (2,0)-form, which is closed, because $\partial^2\eta=\bar\partial\partial\eta=0$. Applying the Poincaré lemma as above, **we obtain that $\partial\eta=d\alpha$, where α is a holomorphic (1,0)-form.**

Step 3: Now, $\partial(\eta-\alpha)=0$, which implies that $\eta-\alpha\in\operatorname{im}\partial$ by Poincaré-Dolbeault-Grothendieck lemma again. Take f such that $\partial f=\eta-\alpha$. Since α is holomorphic, we have $\bar\partial(\eta-\alpha)=\bar\partial\eta=\omega$. **This brings $\omega=\bar\partial\partial f$.** ■

Holomorphic convexity (reminder)

DEFINITION: Let $\Omega \subset \mathbb{C}^n$ be an open subset. It is called **a domain of holomorphy** if for any connected open subset $V \subset \mathbb{C}^n$ such that $W := \Omega \cap V$ is connected, there exists a function $f \in H^0(\mathcal{O}_W)$ which cannoe be extended to V.

DEFINITION: Let $K \subset X$ be a compact set in a complex variety. **A holomorphic hull** of K is

$$\widehat{K} := \left\{ z \in X \;\; | \;\; |f(z)| \leqslant \sup_{z \in K} |f(z)| \;\; \forall f \in \mathcal{O}_X \right\}.$$

DEFINITION: A variety X is called **holomorphically convex** if the holomorphic hull of a compact subset $K \subset X$ is always compact.

Claim 1: A complex variety X **is holomorphically convex if and only if $X = \bigcup_i K_i^\circ$, where $K_0 \subset K_1 \subset ... \subset K_n \subset ...$ is a sequence of holomorphically convex compact subsets and K_i° is the interior of K_i.**

THEOREM: Let $\Omega \subset \mathbb{C}^n$ be an open set. **Then Ω is a domain of holomorphy. $\Leftrightarrow$ Ω is holomorphically convex.**

Pseudoconvexity

DEFINITION: A function $\psi : X \to [-\infty, \infty[$ on a topological space X is called **exhaustion** if all sublevel sets $\psi^{-1}([-\infty, c])$ are compact.

DEFINITION: A complex manifold X is called **weakly pseudoconvex** if X admits a plurisubharmonic exhausting function $\psi : X \to \mathbb{R}$, and **strongly pseudoconvex** if ψ is strictly psh.

THEOREM: Every holomorphically convex manifold X is weakly pseudoconvex.

Proof. Step 1: Let $K_1 \subset K_2 \subset ...$ be a sequence of holomorphically convex compact sets, with $\Omega = \bigcup K_i$ and $K_i \subset K^\circ_{i+1}$ (Claim 1). For each $z \in X$ there exists a unique $i_z \in \mathbb{Z}^+$ such that $z \in K_{i_z} \backslash K_{i_z - 1}$. Then there exists a holomorphic function $g_z \in \mathcal{O}_\Omega$ such that $\sup_{z \in K_{i_z-1}} |g_z(z)| < |g_z(z)|$. Multiplying this function by a constant and taking a power, we may assume that $g_i(z_i) > i_z$, and $\sup_{z \in K_{i_z-1}} |g_z(z)| < \varepsilon$.

Step 2: Using compactness of $L_i := K_i \backslash K^\circ_{i-1}$, and applying the argument of Step 1 to each point of L_i, we obtain a finite collection of holomorphic functions $g_{i,1}, ...g_{i,n_i}$ such that $\sum_{k=1}^{n_i} |g_{i,k}|^2 > i$ on L_i and $\sum |g_{i,1}|^2 < 2^{-i}$ on K_{i-2}.

Step 3: The function $\psi := \sum_i \sum_{k=1}^{n_i} |g_{i,1}|^2$ is by construction psh. This sum converges, because on each K_i this sum is bounded by a geometric progression, and satisfies $\psi_{L_i} > i$, hence it is exhaustion. ■

Pseudoconvex set which is not holomorphically convex

REMARK: Strong pseudoconvexity implies holomorphic convexity (and the Stein property). This is a difficult theorem, due to Cartan, Oka and Grauert.

REMARK: There exists a weakly pseudoconvex manifold which is not holomorphically convex.

EXAMPLE: Let Γ be a free abelian group acting on $\mathbb{C}^2$ and generated by $\gamma_1(z,w) = (z+1, e^{\sqrt{-1}\,\theta_1}w)$ and $\gamma_1(z,w) = (z+\sqrt{-1}\,, e^{\sqrt{-1}\,\theta_2}w)$, where $\theta_i \in \mathbb{R}$. This action is free and properly discontinuous, hence the quotient $X := \mathbb{C}^2/\Gamma$ is a complex manifold, fibered over an elliptic curve $E := \mathbb{C}/(a+b\sqrt{-1}\,)$, $a, b \in \mathbb{Z}$. The function $\psi(w) := |w^2|$ is Γ-invariant and psh, It is exhaustion on each fiber of the projection $\pi : \mathbb{C}^2 \longrightarrow E$. Moreover, $\psi^{-1}([-\infty, c])$ is a product of E and a closed disk of radius $\sqrt{c}$, hence **ψ is exhaustion, and X is weakly pseudoconvex.**

Pseudoconvex set which is not holomorphically convex (2)

CLAIM: Let Γ be a free abelian group acting on $\mathbb{C}^2$ and generated by $\gamma_1(z,w) = (z+1, e^{\sqrt{-1}\,\theta_1}w)$ and $\gamma_1(z,w) = (z+\sqrt{-1}, e^{\sqrt{-1}\,\theta_2}w)$, where $\theta_i \in \mathbb{R}$, and $X := \mathbb{C}^2/\Gamma$ be the complex manifold constructed above. Assume that θ_1 or θ_2 is irrational. **Then $H^0(\mathcal{O}_X) = \text{const}$.** In particular, **$X$ is not holomorphically convex.**

Proof. Step 1: A pullback of a holomorphic function on X to $\mathbb{C}^2$ is a Γ-invariant function $f(z,w)$. For any fixed w_0, $z \mapsto f(z,w_0)$ is doubly periodic and holomorphic; by Liouville theorem, $z \mapsto f(z,w_0)$ is constant. **Therefore, f is independent from z, $f(z,w) = f_0(w)$, where f_0 is a holomorphic function which satisfies $f_0(e^{\sqrt{-1}\,\theta_i}w) = w$, $i = 1,2$.**

Step 2: Taking a Taylor expansion in the origin, $f(w) = \sum a_k w^k$, **we obtain that $a_k = e^{\sqrt{-1}\,k\theta_i}a_k$**; this is impossible when one of θ_i is irrational, hence $a_k = 0$ for all $k > 1$. ■