Complex analytic spaces

lecture 30: Stein manifolds

Misha Verbitsky

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Plurisubharmonic functions (reminder)

DEFINITION: A function f on a complex manifold is called **plurisubharmonic** (or **psh**) if $dd^c f$ is a positive (1,1)-form, and **strictly plurisubharmonic** if $dd^c f$ is a positive definite (and ipso facto Kähler) form.

REMARK: For any plurisubharmonic function f, and any Hermitian form ω, we have $\Delta(f) \geqslant 0$, where Δ is an elliptic operator. Applying the strong maximum principle, we obtain
COROLLARY: A plurisubharmonic function on a manifold **cannot have a local maximum, unless it is constant.** ■

EXAMPLE: A sum of plurisubharmonic functions is plurisubharmonic.
EXAMPLE: Let f be a holomorphic function. Then $dd^c|f|^2 = 2\sqrt{-1}\,\partial\bar\partial f\bar f = 2\sqrt{-1}\,(\partial f \wedge \overline{\partial f})$, hence $|f|^2$ is plurisubharmonic.

COROLLARY: Let $f_1, ..., f_n$ be a collection of holomorphic functions on a complex manifold. **Then $\sum_i |f_i|^2$ is plurisubharmonic, hence it cannot have a maximum.**
EXAMPLE: Let $\mu \in C^\infty\mathbb{R}$. Then

$$dd^c(\mu(f))|_m = \mu'(f(z))^2 dd^c f + \mu''(f(z)) df \wedge d^c f.$$

Therefore, **for any psh function f, the composition $\mu(f)$ is psh when $\mu'' \geqslant 0$ and $\mu' > 0$.**

Regularized maximum

CLAIM: Let $\mu: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth function, monotonous in all arguments and convex, and $\varphi_1, ..., \varphi_n$ a set of plurisubfarmonic functions. **Then $\mu(\varphi_1, ..., \varphi_n)$ is also plurisubharmonic.**

Proof: The same formula

$$dd^c(\mu(f))|_m = \mu'(f(z))^2 dd^c f + \mu''(f(z)) df \wedge d^c f.$$

(appropriately re-conceptualized). ■

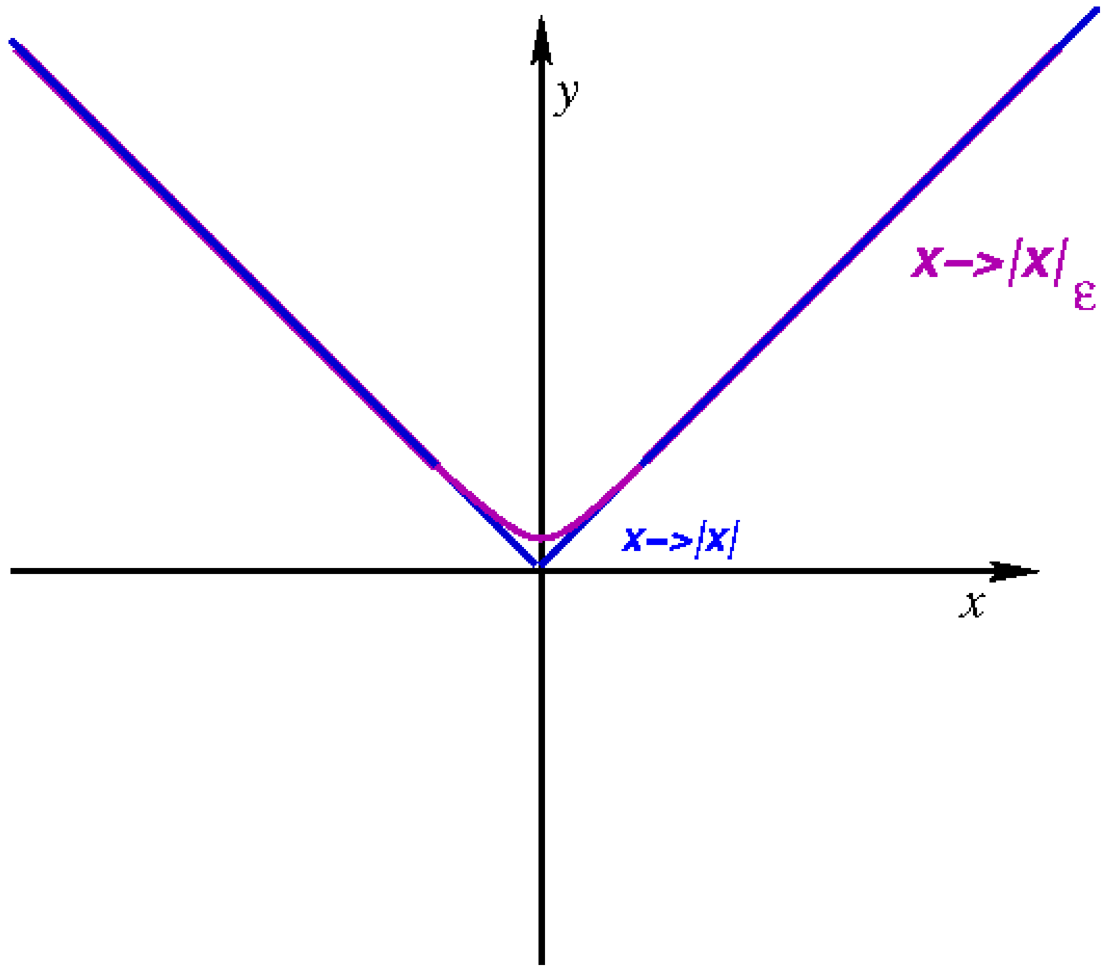
DEFINITION: (Demailly) Let $\mu: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth, convex function, increasing in both arguments. Suppose that for all $|x-y| \geqslant \varepsilon$, one has $\mu(x,y) = \max(x,y)$, and also $\mu(x,y) = \mu(y,x)$, $\mu(y+\alpha, x+\alpha) = \mu(x,y)+\alpha$. Then μ is called **a regularized maximum** and denoted as $\max_\varepsilon(x,y)$.

COROLLARY: A regularized maximum of smooth (strictly) plurisubharmonic functions is smooth and (strictly) psh.

Proof: Apply the previous claim to $\mu(x,y) = \max_\varepsilon(x,y)$. ■

Regularized maximum (2)

EXAMPLE: Take a smooth, convex function $x \mapsto |x|_\varepsilon$ which approximates $x \mapsto |x|$



Then $\max_\varepsilon(x, y) := 1/2|x-y|_\varepsilon + 1/2|x+y|_\varepsilon$ is a regularized maximum **(prove this)**.

Pseudoconvexity (reminder)

DEFINITION: A function $\psi : X \to [-\infty, \infty[$ on a topological space X is called **exhaustion** if all sublevel sets $\psi^{-1}([-\infty, c])$ are compact.

DEFINITION: A complex manifold X is called **weakly pseudoconvex** if X admits a plurisubharmonic exhausting function $\psi : X \to \mathbb{R}$, and **strongly pseudoconvex** if ψ is strictly psh.

THEOREM: Every holomorphically convex manifold X is weakly pseudoconvex.

REMARK: There exists a weakly pseudoconvex manifold which is not holomorphically convex.

Stein manifolds

DEFINITION: Let X be a complex manifold. We say that $\mathcal{O}_X$ **locally separates points** if every point $x \in X$ has a neighbourhood V such that for any $y \in V \backslash x$ there exists a function $f \in H^0(X, \mathcal{O}_X)$ such that $f(x) \neq f(y)$. **A Stein manifold** is a holomorphically convex manifold X such that $\mathcal{O}_X$ locally separate points.

PROPOSITION: Let X be a complex manifold such that $\mathcal{O}_X$ locally separate points. **Then X admits a smooth, non-negative, strictly psh function.**

Proof. Step 1: Let $x \in X$ We start by showing that there exists a smooth, non-negative psh function u_x which is strictly psh in neighbourhood of x. Fix an open set $V \ni x$ such that for all $y \in V \backslash x$ there exists a function $f_y \in H^0(X, \mathcal{O}_X)$ such that $f(x) \neq f(y)$. Rescaling and adding a constant, we may assume that $f(x) = 0, f_y(y) > 1$. Without restricting generality, we may assume that $V \Subset X$. By compactness of ∂V, **we find finitely many functions $f_{y_1}, ..., f_{y_N} \in \mathcal{O}_X$ such that $\psi_x := \sum |f_{y_i}|^2$ satisfies $\psi_x(x) = 0$ and $\sup_{\partial V} \psi_x > 1$.**

Stein manifolds (2)

Step 2: Without restricting the generality, we may assume that V is biholomorphic to an open ball. We denote by $|z|^2$ the standard Kähler potential on V. Let $\varphi_x := \max_\varepsilon(\psi_x, (|z|^2+1)/3)$. This function is plurisubharmonic on V, equal to $(|z|^2+1)/3$ in a neighbourhood of 0, and to ψ_x near the boundary of V; we extend it to X by setting $\varphi_x = \psi_x$ on $X\backslash V$. **The function $\psi_x \in C^\infty X$ is positive, plurisubharmonic, and strictly plurisubharmonic in a neighbourhood U_{x_i} of $x \in X$.**

Step 3: Choose a locally finite covering of X by open balls with centers in x_i, and let φ_{x_i} be the corresponding psh functions constructed in Step 1. Choosing an appropriate sequence of ψ_{x_i}, we can always assume that $X = \bigcup_i U_{x_i}$, where U_{x_i} is a neighbiurhood where ψ_{x_i} is strictly psh. Choose a sequence $\varepsilon_i > 0$ such that the sum $\sum_i \varepsilon_i \varphi_{x_i}$ converges (this is possible to show using the diagonal method; **prove it**). **Then $\sum_i \varepsilon_i \varphi_{x_i}$ is strictly plurisubharmonic.** ■

COROLLARY: Every Stein manifold is strongly pseudoconvex.

Proof: In Lecture 29, we constructed an exhausting psh function Ψ on any holomorphically convex manifold. Let $\Phi \in C^\infty X$ be a positive, strictly psh function constructed in the previous proposition. Then $\Psi + \Phi$ is exhausting and strictly psh. ■

Equivalent definitions of Stein manifolds

The following results are highly non-trivial. I state them without giving or indicating the proof.

REMARK: K. Oka (1942, 1954) has proven that, conversely, **any strongly pseudoconvex complex manifold is Stein.** Then, **could define Stein manifold as Kähler manifold admitting an exhausting Kähler potential.**

REMARK: Clearly, any closed complex submanifold in $\mathbb{C}^n$ is Stein **(prove it)**. The converse statement is also true: **any Stein manifold can be realized as a closed complex submanifold in $\mathbb{C}^n$.** This result is due to Remmert, Narasimhan and Bishop.

REMARK: Another characterization of Stein manifolds is due to H. Cartan: **a complex manifold X is Stein if and only if $H^i(X, F) = 0$ for all $i > 0$ and all coherent sheaves F on X.**

CR-manifolds

Definition: Let M be a smooth manifold, $B \subset TM$ a sub-bundle in a tangent bundle, and $I: B \longrightarrow B$ an endomorphism satisfying $I^2 = -1$. Consider its $\sqrt{-1}$-eigenspace $B^{1,0}(M) \subset B \otimes \mathbb{C} \subset T_C M = TM \otimes \mathbb{C}$. Suppose that $[B^{1,0}, B^{1,0}] \subset B^{1,0}$. Then (B, I) is called **a CR-structure on M**.

Example: A complex manifold is CR, with $B = TM$. Indeed, **$[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$ is equivalent to integrability of the complex structure** (Newlander-Nirenberg).

Example: Let X be a complex manifold, and $M \subset X$ a hypersurface. Then $B := \dim_{\mathbb{C}} TM \cap I(TM) = \dim_{\mathbb{C}} X - 1$, hence $\operatorname{rk} B = n - 1$. Since $[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$, **$M$ is a CR-manifold.**

Definition: **The Frobenius form** of a CR-manifold is the tensor $B \otimes B \longrightarrow TM/B$ mapping X, Y to the projection $\Pi_{TM/B}([X, Y])$. It is an obstruction to integrability of the foliation given by B.

Contact CR-manifolds.

Complex algebraic geometry is a rich source of contact structures.

Definition: Let (M,B,I) be a CR-manifold, with $\operatorname{codim} B = 1$. Then M is called **a contact CR-manifold** if its Frobenius form is non-degenerate.

Remark: Since $[B^{1,0},B^{1,0}] \subset B^{1,0}$ and $[B^{0,1},B^{0,1}] \subset B^{0,1}$, the Frobenius form is a pairing between $B^{0,1}$ and $B^{1,0}$. **This means that it is Hermitian.**

DEFINITION: This Hermitian form is called **Levi form** of a CR-manifold.

Definition: Let (M,B,I) be a CR-manifold, with $\operatorname{codim} B = 1$. Then M is called **a strictly pseudoconvex CR-manifold** if its Levi form is positive definite.

Example: Let h be a function on a complex manifold such that $\partial\bar\partial h = \omega$ is a positive definite Hermitian form, and $X = h^{-1}(c)$ its level set. Then the Frobenius form of X is equal to $\omega|_X$ (see the next slide). In particular, X **is a strictly pseudoconvex CR-manifold**.

CR-manifolds and plurisubharmonic functions.

PROPOSITION: Let M be a complex manifold, $\varphi \in C^\infty M$ a smooth function, and s a regular value of φ. Consider $S := \varphi^{-1}(s)$ as a CR-manifold, with $B = TS \cap I(TS)$ and let Φ be its Levi form, taking values in

$$TS/B = \ker d\varphi / \ker d\varphi \cap I(\ker d\varphi)$$

Then $d^c\varphi : TS/B \longrightarrow C^\infty S$ trivializes TS/B. Consider tangent vectors $u, v \in B_x S$. **Then $-d^c\varphi(\Phi(u,v)) = dd^c\varphi(x,y))$.**

Proof: Extend u, v to vector fields $u, v \in B = \ker d\varphi \cap I(\ker d\varphi)$. Then $-d^c\varphi(\Phi(u,v)) = -d^c\varphi([u,v]) = dd^c\varphi(u,v)$. ∎

COROLLARY: Let M be a complex manifold, $\varphi \in C^\infty M$ a strictly plurisubharmonic function, and s a regular value of φ. **Then $S := \varphi^{-1}(s)$ is strictly pseudoconvex.**

Proof: By the above proposition, the Levi form of S is expressed as $dd^c\varphi(u,v)$, hence it is positive definite. ∎

Geometry of strictly pseudoconvex CR-manifolds

The following results are highly non-trivial. I state them without giving or indicating the proof.

THEOREM: (Grauert's solution of the Levi problem) Let M be a complex manifold with smooth boundary S. Assume that the Levi form on S is strictly pseudoconvex. **Then M is holomorphically convex.**

DEFINITION: Let (S, B, I) be a CR-manifold. A function f on S is called **CR-holomorphic** if for any vector field $v \in B^{0,1}$, we have $\operatorname{Lie}_v f = 0$.

THEOREM: (Rossi-Andreotti-Siu)
Let S be a compact strictly pseudoconvex CR-manifold, $\dim_{\mathbb{R}} S \geqslant 5$, and $H^0(\mathcal{O}_S)_b$ the ring of bounded CR-holomorphic functions. **Then S is a boundary of a Stein manifold M with isolated singularities,** such that $H^0(\mathcal{O}_S)_b = H^0(\mathcal{O}_M)_b$, where $H^0(\mathcal{O}_M)_b$ denotes the ring of bounded holomorphic functions. Moreover, M is defined uniquely, $M = \operatorname{Spec}(H^0(\mathcal{O}_S)_b)$.

THEOREM: (Dan Burns, John M. Lee)
Let S be a compact strictly pseudoconvex CR-manifold, and $\operatorname{Aut}_0(S)$ the connected component of its group of automorphisms. **Then $\operatorname{Aut}_0(S)$ is compact** unless S is equivalent to the standard sphere $S^{2n-1} \subset \mathbb{C}^n$ with its induced CR-structure. In the latter case $\operatorname{Aut}_0(S) = U(1, n)$.