Complex analytic spaces 1, written test: Hodge decomposition

Rules: This is a written test for the next week. Please write your solutions and bring me no later than the Monday class the week after. We shall discuss this test in class afterwards.

Exercise 1.1. Let $\omega_1, \omega_2, \omega_3$ be 2-forms on $V := \mathbb{R}^4$, and Vol its volume form. Assume that the matrix $A_{ij} := \frac{\omega_i \wedge \omega_j}{\mathrm{Vol}}$ is positive definite. Prove that there exists an operator $I \in \mathrm{End}(V)$ such that $I^2 = -\mathsf{Id}_V$ and the action of I on $\Lambda^*(V)$ preserves the 3-dimensional subspace $\langle \omega_1, \omega_2, \omega_3 \rangle \subset \Lambda^2 V$.

Exercise 1.2. Let $\omega_1, \omega_2, \omega_3 \in \Lambda^{1,1}(\mathbb{C}^2)$ be real 2-forms which satisfy $\omega_i \wedge \omega_i = 0$. Assume that the 4-forms $\omega_1 \wedge \omega_2$ and $\omega_2 \wedge \omega_3$ are proportional to the volume form of $\mathbb{C}^2 = \mathbb{R}^4$ with positive coefficient. Prove that $\omega_1 \wedge \omega_2$ and $\omega_1 \wedge \omega_3$ are proportional with non-negative coefficient.

Exercise 1.3. Let $\theta \in \Lambda^{1,0} M$ be a (1,0)-form with compact support on a non-compact connected complex manifold (M, I). Assume that θ is closed. Prove that $\theta = 0$.

Exercise 1.4. Let (M, I) be an almost complex manifold, $\dim_{\mathbb{R}} M = 2n$, and $\theta_1, ..., \theta_n \in \Lambda^{1,0}(M)$ pointwise linearly independent, closed 1-forms. Prove that the almost complex structure I is integrable, not using Newlander-Nirenberg theorem.

Exercise 1.5. Consider the natural $U(n+1)$-action on $\mathbb{C}P^n$. Prove that there exists a unique (up to a constant multiplier) $U(n+1)$-invariant symplectic form $\omega \in \Lambda^2(\mathbb{C}P^n)$. Prove that this form is of Hodge type (1,1).

Hint. Use Schur's lemma.

Remark 1.1. This 2-form is called **the Fubini-Study form**, and the corresponding Riemannian metric **the Fubini-Study metric**.

Exercise 1.6. Let $\phi: \mathbb{C}^{n+1} \backslash 0 \longrightarrow \mathbb{C}P^n$ be a natural projection. Its fibers are identified with $\mathbb{C} \backslash 0$; we denote the linear holomorphic coordinate on these fibers by z. Let $f \in C^\infty(\mathbb{C}^{n+1} \backslash 0)$ be a positive real-valued function which is proportional to $|z|^2$ on each fiber of ϕ. Prove that $\partial\bar\partial(\log f) = \phi^* \eta_f$ for some 2-form $\eta_f \in \Lambda^{1,1}(\mathbb{C}P^n)$. Prove that $-\sqrt{-1}\eta_f$ is the Fubini-Study form when $f(z_1, ..., z_{n+1}) = \sum_{i=1}^{n+1} |z_i|^2$.