Complex varieties 3, written test: Meromorphic functions

Rules: This is a written test for the next week. Please write your solutions and bring me no later than the Monday class the week after. We shall discuss this test in class afterwards.

Exercise 3.1. Let $X \longrightarrow \mathbb{C}P^1$ be a holomorphic map, written in affine coordinates as $x \mapsto (g : f)$, where f and g are holomorphic functions, defined locally on X. By definition, the function $\mu(x) := \frac{f}{g}$ is meromorphic.

a. Prove that any meromorphic function on a 1-dimensional complex manifold can be obtained this way.

b. Construct a meromorphic function $\mu: \mathbb{C}^2 \longrightarrow \mathbb{C}$ which cannot be associated with any holomorphic map $\mathbb{C}^2 \longrightarrow \mathbb{C}P^1$ in this fashion.

Exercise 3.2. Let μ be a meromorphic function on a complex manifold X, $P \subset X$ its pole set, and $\Gamma_\mu \subset (X \backslash P) \times \mathbb{C}$ the graph of the holomorphic function $\mu\Big|_{X \backslash P}$. Prove that the closure of Γ_μ in $X \times \mathbb{C}P^1$ is a complex subvariety.

Hint. Use the Remmert-Stein theorem.

Exercise 3.3. Let f be a holomorphic function on $X \backslash D$, where D is a divisor on a complex manifold X. Assume that for any sequence $\{x_i\} \subset X$ converging to D we have $\lim_i |f(x_i)| = \infty$. Prove that f is meromorphic.

Exercise 3.4. Consider the action of the dihedral group D_n on $\mathbb{C}P^1$ generated by the maps $x : y \mapsto x : \varepsilon y$, where ε is an n-th root of unity, and $x : y \mapsto y : x$. Prove that $\frac{\mathbb{C}P^1}{D_n} = \mathbb{C}P^1$.