Teoria Ergódica Diferenciável, assignment 2: Radon-Nikodym theorem

Rules: This is a class assignment for August 28. Please try to write the solutions in class at August 28 and give them to your monitor Ermerson Rocha Araujo. No-one is penalized for failing to write the solutions, but being good at assignments would simplify getting good grades at your exams.

2.1 Hahn decomposition

Definition 2.1. Let S be a space with σ-algebra $\mathfrak{U} \subset 2^S$. **Signed measure** is a σ-aditive function $\mathfrak{U} \longrightarrow \mathbb{R}$.

Exercise 2.1. Let $\rho : \mathfrak{U} \longrightarrow \mathbb{R}$ be a signed measure.

a. Let $X_i \in \mathfrak{U}$ be a sequence of sets such that $\lim_i \rho(X_i) = -\infty$. Consider $Y_n := \bigcup_{i=0}^n X_i$. Prove that either $\lim_n \rho(Y_n) = -\infty$ or $\limsup_n \rho(X_n \backslash Y_{n-1}) = \infty$. In the first case prove that $\rho(\bigcup Y_i) = -\infty$.

b. Let $\beta := \inf_{B \in \mathfrak{U}} \rho(B)$. If $\beta = -\infty$, prove that a union $\bigcup Z_{n_i}$ of some $Z_n := X_n \backslash Y_{n-1}$ defined above satisfies $\rho(\bigcup Z_{n_i}) = \infty$. Deduce that $\beta > -\infty$.

c. Let $E_1, E_2 \in \mathfrak{U}$ be elements such that $\rho(E_i) \leqslant \beta + \varepsilon_i$. Prove that $\rho(E_1 \triangle E_2) \geqslant -\varepsilon_1 - \varepsilon_2$. Deduce that $\rho(E_1 \cup E_2) \leqslant \beta + \varepsilon_1 + \varepsilon_2$ and $\rho(E_1 \cap E_2) \leqslant \beta + \varepsilon_1 + \varepsilon_2$.

d. Let $\{E_i\}$ be a sequence such that $\rho(E_i) \leqslant \beta + \frac{1}{2^i}$. Prove that $B_j := \cup_{i>j} E_i$ satisfies $\lim_i \rho(B_i) = -\beta$.

e. In these assumptions, prove that $\rho(B) = \beta$, where $B = \bigcap B_i$.

Hint. Show that $|\rho(B_j \backslash E_j)| \leqslant \frac{1}{2^{j-1}}$, $\rho(B_j \backslash B_{i+j}) \leqslant \frac{1}{2^{j-3}}$, and $\rho(B_j \backslash B) \leqslant \frac{1}{2^{j-4}}$.

Exercise 2.2. In assumptions of the previous exercise, let $A := S \backslash B$. Prove that $\rho(X) \geqslant 0$ for each $X \subset A$ and $\rho(X) \leqslant 0$ for each $X \subset B$.

Definition 2.2. Let ρ be a signed measure on S. A measurable set $X \subset S$ is called **ρ-negligible** if for any $X_1 \subset X$, one has $\rho(X_1) = 0$.

Exercise 2.3. Prove that ρ is a measure on A, $-\rho$ is a measure on B, and the decomposition $S = A \coprod B$ is defined uniquely up to a ρ-negligible set.

Definition 2.3. This decomposition is called **the Hahn decomposition** of the signed measure ρ.

2.2 Absolutely continuous measures

Definition 2.4. Let S be a space equipped with a σ-algebra, and μ, ν two measures on this σ-algebra. We say that ν is **absolutely continuous** with respect to μ, denoted by $\nu \ll \mu$, if for each measurable set A, $\mu(A) = 0$ implies $\nu(A) = 0$.

Exercise 2.4. Let $\nu \ll \mu$ be non-zero finite measures on a space S with a σ-algebra. Consider the signed measure $\nu - \varepsilon\mu$, where $\varepsilon > 0$, and let $S = A_\varepsilon \coprod B_\varepsilon$ be its Hahn decomposition.

a. Prove that $\nu(B_\varepsilon) \leqslant \varepsilon\mu(S)$ and $\lim\limits_{\varepsilon \longrightarrow 0} \nu(B_\varepsilon) = 0$.

b. Deduce from this that $\nu(A_\varepsilon) > \frac{1}{2}\nu(S)$ for ε sufficiently small.

c. Prove that $\mu(A_\varepsilon) > 0$ for ε sufficiently small.

Exercise 2.5. $\nu \ll \mu$ be finite measures on a space S with a σ-algebra, with $\nu \neq 0$. Prove that there exists a non-negative measurable function f, positive on a set of positive measure, such that $\nu - f\mu \geqslant 0$.

Hint. Apply the Hahn decomposition to $\nu - \varepsilon\mu$ and take $f = \varepsilon\chi_{A_\varepsilon}$

Exercise 2.6. (Lebesgue's monotone convergence theorem) Let $\{f_i\}$ be monotone sequence of integrable functions on a space S with measure, universally bounded in L^1-norm. Prove that $\{f_i\}$ converges in L^1-norm to its pointwise limit.

Exercise 2.7. Let M be a space with finite measure, and $\{f_\alpha\}$ a countable set of positive measurable functions, universally bounded in L^1-norm. Prove that $\sup_\alpha f_\alpha$ is measurable.

Hint. Use Lebesgue's monotone convergence theorem.

Exercise 2.8. (Radon-Nikodym theorem) Let $\nu \ll \mu$ be finite measures on a space S with σ-algebra. Denote by C the infimum $C := \inf_f \int_S \nu - f\mu$, where f is a non-negative measurable function, such that $\nu - f\mu$ is non-negative. Consider a sequence $\{f_i\}$ of non-negative measurable functions such that $f_i\mu \leqslant \nu$ and $\lim_i \int_S \nu - f_i\mu = C$. Prove that $C = 0$ and $f := \sup_{f_\alpha \in \mathcal{F}} f_i$ satisfies $f\mu = \nu$.

Hint. Otherwise consider the measure $\nu_1 := \nu - f\mu$, prove that $\nu_1 \ll \mu$ and find a non-negative measurable function g, positive on a set of positive measure, such that $\nu_1 - g\mu \geqslant 0$. Then $f + g$.

2.3 Supplement by Ermerson Rocha Araujo

Exercise 2.9. Find a continuous transformation $f : X \to X$ of a compact metric space and an **infinite** measure μ in X for which the statement of Poincar's Recurrence Theorem does not hold.

Exercise 2.10. Let X be a compact metric space and $f : X \to X$ a continuous map. If E is a subset of X, we put

$$\tau(E, x) = \limsup_{n\to\infty} \frac{1}{n} \#\{0 \leqslant j < n \ : \ f^j(x) \in E\}.$$

Suppose that for every subset $U \subset X$ and every $x \in X$ there exists $\mu \in \mathcal{M}_{prob}(X, f)$ such that $\mu(U) \leqslant \tau(U, x)$.

a. We say that a compact set $\Lambda \subset X$ is a center of attraction for f if $f(\Lambda) \subset \Lambda$ and $\tau(U, x) = 1$ for every $x \in X$ and every neighborhood U of Λ. If Λ is a center of attraction and contains no proper subset with the same property, it is called a minimal center of attraction. Prove that there exists a unique minimal center of attraction Λ for f, and that $f(\Lambda) = \Lambda$.

b. If S is closure of the union of the supports of all probability measures $\mu \in \mathcal{M}_{prob}(X, f)$, then $\Lambda = S$

Exercise 2.11. Let X be a set, $\mathcal{A}$ a σ-algebra on X and $T : X \to X$ a measurable map. If $\mu_i \in \mathcal{M}_{prob}(X, f)$, $i = 1, \dots, n$ are ergodic and μ_i is not absolutely continuous with respect to μ_j for $i \neq j$, prove that there exist disjoint sets $A_i \in \mathcal{A}$ $i = 1, \dots, n$, such that

$$\bigcup_{i=1}^{n} A_i = X,$$

$$\mu_i(A_j) = \delta_{ij}.$$

The result still holds if we have infinite countable $\mu_i's$?

Hint. Use Birkhoff's Ergodic Theorem.