Teoria Ergódica Diferenciável, assignment 3: Tychonoff theorem

Rules: This is a class assignment for September 6. Please try to write the solutions in class at September 6 and give them to your monitor Ermerson Rocha Araujo. No-one is penalized for failing to write the solutions, but being good at assignments would simplify getting good grades at your exams.

Remark 3.1. All topological spaces in this assignment are assumed to be Hausdorff.

Definition 3.1. **An open cover**, or **a cover** of a topological space M is a collection $\{U_\alpha\}$ of non-empty open subsets $U_\alpha \subset M$ such that $\bigcup_\alpha U_\alpha = M$. **A subcover** is a subset $\{U_\beta\} \subset \{U_\alpha\}$ such that $\bigcup_\beta U_\beta = M$. **Compact** is a topological space such that any of its open covers has a finite subcover.

Definition 3.2. **Subbase** of topology on a topological space M is a collection of open sets U_α such that any open set can be obtained as an infinite union of finite intersections of $\{U_\alpha\}$.

Exercise 3.1. Let $f: M \longrightarrow N$ be a continuous bijective map of topological spaces, with M compact. Prove that f is a homeomorphism.

Definition 3.3. Let $\prod_\alpha M_\alpha$ be a product of topological spaces M_α. **Product topology**, or **Tychonoff topology** is topology for which the subbase is all the sets of the form $\pi_a^{-1}(U)$, where $\pi_\alpha: \prod_\alpha M_\alpha \longrightarrow M_\alpha$ is a projection to the M_α-component, and $U \subset M_\alpha$ is an open set.

Exercise 3.2. Prove that a countable product of metrizable compacts is compact.

Exercise 3.3. Let $\mathcal{V}$ is a set of open coverings of a topological space M, such that for any two coverings $\{V_\alpha\}, \{W_\beta\} \in \mathcal{V}$, either $\{V_\alpha\} \subset \{W_\beta\}$ or $\{W_\beta\} \subset \{V_\alpha\}$. Assume that none of the covers $\{V_\alpha\} \in \mathcal{V}$ has a finite subcover. Prove that the union of all $\{V_\alpha\} \in \mathcal{V}$ has no finite subcovers.

Exercise 3.4. Let X be a non-compact topogical space. Prove that M has a cover $\{V_\alpha\}$ such that $\{V_\alpha\}$ has no finite subcovers, but any strictly bigger open cover $\{V_\alpha\} \subsetneq \{W_\beta\}$ has a finite subcover.

Hint. Use Zorn lemma and the previous exercise.

Definition 3.4. We shall call such covers **maximal**.

Exercise 3.5. Let $\{V_\alpha\}$ be a maximal cover of a non-compact topological space M.

a. Prove that for any two open sets $U_1, U_2 \notin \{V_\alpha\}$, the intersection $U_1 \cap U_2$ does not belong to $\{V_\alpha\}$.

b. Prove that an intersection of open subsets $U_1, U_2, ..., U_n \notin \{V_\alpha\}$ also does not belong to $\{V_\alpha\}$.

c. Let $U = \bigcap_{i=0}^n U_i$. Assume that $U \in \{V_\alpha\}$. Prove that at least one of U_i belongs to $\{V_\alpha\}$.

Exercise 3.6. Let M be a non-compact topological space, and $\{V_\alpha\}$ its maximal open cover. Consider a subbase $\mathcal{R}$ of topology on M. Prove that $\{V_\alpha\}$ contains a subcover $\{W_\beta\} \subsetneq \{V_\alpha\}$ with all $W_\beta \in \mathcal{R}$.

Hint. Use the previous exercise.

Exercise 3.7. Prove **Alexander subbase theorem:** Let $\mathcal{S}$ be a subbase of topology on a topological space M. Suppose that any cover of M by elements of $\mathcal{S}$ contains a finite subcover. Prove that M is compact.

Remark 3.2. Alexander subbase theorem is equivalent to Axiom of Choice.

Definition 3.5. **Minimal cover** is a cover which has no proper subcovers.

Exercise 3.8. Let $M = \prod_\alpha M_\alpha$ be a product of topological spaces with product topology, and $\mathcal{S}$ the subbase of topology on M consisting of all sets of the form $\pi_a^{-1}(U)$, where $\pi_\alpha : \prod_\alpha M_\alpha \longrightarrow M_\alpha$ is a projection to the M_α-component, and $U \subset M_\alpha$ is open. Prove that for all minimal cover $\{V_\mu\}$ of M by elements of $\mathcal{S}$ there exists α and a minimal cover $\{U_\tau\}$ of M_α such that $\{V_\mu\} = \{\pi_\alpha^{-1}(U_\tau)\}$.

Exercise 3.9. Let $M = \prod_\alpha M_\alpha$ be a product of topological spaces with product topology, and $\mathcal{S}$ the subbase of topology defined above. Suppose that all M_α are compact. Prove that any cover $\{V_\mu\}$ of M by elements of $\mathcal{S}$ has a finite subcover.

Exercise 3.10. Prove **Tychonoff theorem:** a product of any family of compact spaces is compact.

Hint. Use the previous exercise and Alexander subbase theorem

3.1 Supplement by Ermerson Rocha Araujo

Exercise 3.11. Let an invertible topological dynamical system $f : X \to X$ with X finite and endowed with the discrete topology.

a. Show that there is a **unique** f-invariant Borel probability measure on (X, f) if and only if (X, f) consists of a single periodic orbit.

b. More generally, if μ is a f-invariant Borel probability measure on (X, f), then μ can be written in a unique way as a convex combination $\sum_{i=1}^{n} \alpha_i \mu_i$ where μ_i is ergodic for each i.

Exercise 3.12. Consider the sequence $(2^n)_{n\in\mathbb{N}}$ of powers of two. Let any positive integer $a = a_1 a_2 \cdots a_l$, $a_j \in \{0, 1, 2, \ldots, 9\}$. Prove that there are infinitely many n such that decimal expansion of 2^n starts with a.

Hint. Show that an integer m starts with $a_1 a_2 \cdots a_l$ if and only if there exists $k \in \mathbb{N}$ such that $a_1.a_2 \cdots a_l \times 10^k \leqslant m < a_1.a_2 \cdots (a_l + 1) \times 10^k$.

Exercise 3.13. Let $f : X \to X$ a measurable map and μ, ν f-invariant measures such that $\mu \neq \nu$. Show that the measure $\eta = \dfrac{1}{2}(\mu + \nu)$ is **not** ergodic.

Hint. A set E with $\eta(E) = 1$ must have $\mu(E) = 1$ and $\nu(E) = 1$.