Teoria Ergódica Diferenciável, assignment 6: Regular measures

Rules: This is a class assignment for September 27. Please try to write the solutions in class at September 27 and give them to your monitor Ermerson Rocha Araujo. No-one is penalized for failing to write the solutions, but being good at assignments would simplify getting good grades at your exams.

Definition 6.1. Let M be a locally compact topological space. **Borel algebra** is a σ-algebra $\mathbf{S}$ of subsets of M generated by compact subsets. **Borel measure** is a measure on $(M, \mathbf{S})$.

Exercise 6.1. Topological space M is called **σ-compact** if any closed subset can be obtained as a countable union of compact subsets. Prove that for such M the Borel σ-algebra $\mathbf{S}$ is generated by open sets.

Further on, we shall always assume that the topological space M is Hausdorff and σ-compact.

Definition 6.2. Let $\mathbf{C}$ be the set of compact subsets of M. A function $\lambda : \mathbf{C} \longrightarrow \mathbb{R}^{\geqslant 0}$ is

a. **Monotone**, if $\lambda(A) \leqslant \lambda(B)$ for $A \subset B$

b. **Additive**, if $\lambda(A \coprod B) = \lambda(A) + \lambda(B)$

c. **Semiadditive**, if $\lambda(A \cup B) \leqslant \lambda(A) + \lambda(B)$

In this case λ is called **volume function**.

Definition 6.3. Let λ be a volume on M. For any $S \subset M$, define **inner measure** $\lambda_*(S) := \sup\limits_C \lambda(C)$, where supremum is taken over all compact $C \subset S$, and **outer measure** $\lambda^*(S) := \inf\limits_U \lambda_*(U)$, where infimum is taken over all open $U \supset S$.

Exercise 6.2. Prove that for any open subset $U \subset M$, one has $\lambda_*(U) = \lambda^*(U)$.

Exercise 6.3. Prove that Lebesgue measure defines a volume on $M = \mathbb{R}^n$, and in this situation $\lambda^*(C) = \lambda(C)$ for any compact subset $C \subset M$.

Exercise 6.4. Let $U, V \subset M$ be open subsets, and $C \subset U \cup V$ a compact subset. Prove that there exist compact subsets $C_U \subset U, C_V \subset V$, such that $C_U \cup C_V = C$.

Hint. Prove that any two non-intersecting compact subsets of a Hausdorff topological space have non-intersecting open neighbourhoods. Use this to find non-intersecting open neighbourhoods $U_1 \supset C\backslash V$ and $V_1 \supset C\backslash U$. Take $C_V := C\backslash U_1$, $C_U := C\backslash V_1 \subset U$.

Exercise 6.5. Let M be a topological space equipped with the volume $\lambda : \mathbf{C} \longrightarrow \mathbb{R}^{\geqslant 0}$, and $U, V \subset M$ open subsets. Prove that $\lambda_*(U \cup V) \leqslant \lambda_*(U) + \lambda_*(V)$

Hint. Use the previous exercise.

Exercise 6.6. Prove that λ^* is **semiadditive**, that is, satisfies $\lambda^*(\bigcup A_i) \leqslant \sum_i \lambda^*(A_i)$ for any finite collection of subsets $A_i \subset M$.

Hint. Use the previous exercise.

Exercise 6.7. Prove that λ^* is **countably semiadditive**, that is, satisfies $\lambda^*(\bigcup A_i) \leqslant \sum_i \lambda^*(A_i)$ for any countable collection of subsets $A_i \subset M$.

Hint. Reduce everything to the case $\lambda_*(\bigcup_{i=0}^{\infty} U_i) \leqslant \sum \lambda^*(U_i)$, where U_i are open. For any compact $C \subset \bigcup U_i$, it is covered by only finite set of U_i. Use this to prove that

$$\lambda^*\left(\bigcup_{i=1}^{\infty} U_i\right) \leqslant \lim_n \lambda^*(U_1 \cup U_2 \cup ...U_n).$$

Exercise 6.8. Let $U \subset M$ be open, C compact, and $D \subset U\backslash C$ compact.

a. Prove that $\lambda^*(C \cap U) \leqslant \lambda_*(U\backslash D) = \sup \lambda(E)$, where supremum is taken over all compacts $E \subset U\backslash D$.

   Hint. $U\backslash D$ is a neighbourhood of $C \cap U$.

b. Prove that

$$\lambda^*(U\backslash C)+\lambda^*(U\cap C) = \sup_D \lambda(D)+\lambda^*(U\cap C) \leqslant \sup_{D,E} \lambda(E)+\lambda(D) = \sup_{D,E} \lambda(D\cup E) \leqslant \lambda^*(U)$$

   Hint. D and E are compact, non-intersecting subsets of U.

Exercise 6.9. Let $U \subset M$ be open, C compact. Prove that

$$\lambda^*(U) = \lambda^*(U\backslash C) + \lambda^*(U \cap C).$$

Hint. Use the previous exercise.

Definition 6.4. Let λ be a volume on M. A subset $A \subset M$ is called **λ^*-measurable**, or **Caratheodory measurable**, if

$$\lambda^*(B) = \lambda^*(B \backslash A) + \lambda^*(B \cap A)$$

for any subset $B \subset M$.

Exercise 6.10. Suppose that

$$\lambda^*(U) = \lambda^*(U \backslash A) + \lambda^*(U \cap A) \tag{6.1}$$

for any open subset $U \subset M$. Prove that A is λ^*-measurable.

Hint. By definition, $\lambda^*(B) = \inf \lambda^*(V)$, where infimum is taken over all open neighbourhoods $V \supset B$. Then (6.1) implies

$$\begin{aligned} \lambda^*(B) =& \inf_V \lambda^*(V) = \inf_V \left(\lambda^*(V \backslash A) + \lambda^*(V \cap A)\right) \\ \geqslant& \lambda^*(V \backslash A) + \lambda^*(V \cap A) \geqslant \lambda^*(B \backslash A) + \lambda^*(B \cap A). \end{aligned}$$

Converse inequality follows from semiadditivity.

Exercise 6.11. Prove that all compact sets are λ^*-measurable.

Hint. Use Exercise 6.9.

Exercise 6.12. Prove that intersection, union, complement of λ^*-measurable sets is λ^*-measurable.

Hint. For any non-intersecting X, Y, the equality $\lambda^*(X \coprod Y) = \lambda^*(X) + \lambda^*(Y)$ implies $\lambda^*(X \coprod Y) = \lambda^*(X \backslash A) + \lambda^*(X \cap A) + \lambda^*(Y \backslash A) + \lambda^*(Y \cap A)$ for any λ^*-measurable set A.

Exercise 6.13. Let $A = \coprod_{i=1}^\infty A_i$ be a countable union of non-intersecting λ^*-measurable sets. Suppose that for any X with $\lambda^*(X) < \infty$, we have $\lim_N \lambda^*(X \cap \coprod_{i=N}^\infty A_i) = 0$. Prove that A is λ^*-measurable.

Hint. Prove that

$$\lambda^*(A \cap X) = \lambda^*\left(X \cap \coprod_{i=1}^{N-1} A_i\right) + \lambda^*\left(X \cap \coprod_{i=N}^{\infty} A_i\right), \quad \text{and} \quad \lambda^*(X \backslash A) \leqslant \lambda^*\left(X \backslash \left(\coprod_{i=1}^{N-1} A_i\right)\right),$$

and deduce

$$\lambda^*(A \cap X) + \lambda^*(X \backslash A) \leqslant \lambda^*(X) + \lambda^*\left(X \cap \coprod_{i=N}^{\infty} A_i\right).$$

Exercise 6.14. Let X be λ^*-measurable, and $\lambda^*(X \cap A) < \infty$, where $A = \coprod_{i=1}^\infty A_i$ is a countable union of non-intersecting λ^*-measurable sets. Prove that $\lim_N \lambda^*(X \cap \coprod_{i=N}^\infty A_i) = 0$.

Hint. Replacing X by $X \cap A$ and A_i by $A_i \cap X$, we can assume that $X = A$ and $\lambda^*(X) < \infty$. Let $U_i \supset A_i$ be open neighbouhoods satisfying $\lambda^*(U_i) \leqslant \lambda^*(A_i) + \frac{1}{2^i}\varepsilon$. Prove that $\lambda^*(U_i \backslash A_i) \leqslant \frac{1}{2^i}\varepsilon$. Find a compact K in $\bigcup_i U_i$ such that $\lambda^*(A) = \lambda(K) - \varepsilon$, and find N such that $K \subset \bigcup_{i=0}^N U_i$. Prove

$$\lambda^*(A) = \lambda(K) - \varepsilon \leqslant \lambda^*\left(\bigcup_{i=1}^N U_i\right) \leqslant \sum_{i=1}^N \lambda^*(A_i) + \lambda^*(U_i \backslash A_i) \leqslant \sum_{i=1}^N \lambda^*(A_i) + \varepsilon.$$

Using $\lambda^*(A) = \sum_{i=1}^N \lambda^*(A_i) + \lambda^*\left(\coprod_{i=N}^\infty A_i\right)$, show that $\lambda^*\left(\coprod_{i=N}^\infty A_i\right) \leqslant 2\varepsilon$.

Exercise 6.15. Prove that a countable union of λ^*-measurable sets can be represented as a countable union of non-intersecting λ^*-measurable sets.

Exercise 6.16.

a. Prove that a countable union of λ^*-measurable sets is λ^*-measurable.

Hint. Replace $\bigcup A_i$ by a union of non-intersecting λ^*-measurable sets and apply exercises 6.13 and 6.14.

b. Prove that any Borel set is λ^*-measurable.

c. Prove that λ^* defines a measure on the σ-algebra of Borel sets.

Exercise 6.17. Let $\lambda(K)$ be a Lebesque measure of a compact $K \subset \mathbb{R}^n$. Prove that Lebesgue measurable subset of $\mathbb{R}^n$ is Caratheodory measurable, and the Lebesque measure is equal to λ^*.

Definition 6.5. Radon measure. or **regular measure** on a locally compact topological space M is a Borel measure μ which satisfies the following assumptions.

1. μ is finite on all compact sets.
2. For any Borel set E, one has $\mu(E) = \inf \mu(U)$, where infimum is taken over all open U containing E.
3. For any open set E, one has $\mu(E) = \sup \mu(K)$, where infimum is taken over all compact K contained in E.

Remark 6.1. We have just proved that Lebesgue measure is regular.