Teoria Ergódica Diferenciável, assignment 10: uniquely ergodic measures

Rules: This is a home assignment for November 22. Please bring me the solutions no later than November 29.

Exercise 10.1. Let M be a compact metric space, $T: M \longrightarrow M$ a uniquely ergodic continuous map, and μ a T-invariant probability measure. Prove that all orbits of T are dense in M, or find a counter-example.

Exercise 10.2. Let M be a compact metric space, and $T: M \longrightarrow M$ a continuous map. Assume the sequence $\frac{1}{n}\sum_{i=0}^{n-1} T^i f$ uniformly converges for any continuous f.

a. Prove that T is uniquely ergodic, or find a counterexample.

b. Prove that T is uniquely ergodic if, in addition, it has a dense orbit.

Exercise 10.3. Let $\bar{\Delta}$ be the union of a hyperbolic disk with its absolute, and $T: \Delta \longrightarrow \Delta$ an isometry which has no fixed points in Δ, and only one fixed point in the absolute. Prove that it is uniquely ergodic, or find a counterexample.

Exercise 10.4. Let $k = 1, 2, 3, ..., 9$ be a digit, and $p_k(n)$ be the number of powers of 2 from 1 to 2^n with digital expansion starting from k. Prove that $\lim_n \frac{p_k(n)}{n} = \log_{10}\left(\frac{k+1}{k}\right)$.

Hint. Prove that 2^n starts with k if and only if $n \log_{10}(2) \mod \mathbb{Z}$ belongs to the interval $[\log_{10}(k), \log_{10}(k+1)]$, and apply unique ergodicity.