Teoria Ergódica Diferenciável

lecture 1: spaces with measure

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Boolean algebras

I will start with a brief formal treatment of measure theory. **I assume that the students know the measure theory well enough.**

DEFINITION: The set of subsets of X is denoted by 2^X. **Boolean algebra of subsets if X** is a subset of 2^X closed under **boolean operations** of intersection and complement,

EXERCISE: Prove that the rest of logical operations, such as union and symmetric difference **can be expressed through intersection and the complement.**

REMARK: The Boolean algebras can be defined axiomatically through the axioms called **de Morgan's Laws**. Realization of a Boolean algebra as a subset of 2^X is called **an exact representation**. Existence of an exact representation for any given Boolean algebra is a non-trivial theorem, called **Moore's representation theorem**.

σ-algebras and measures

DEFINITION: Let M be a set **A σ-algebra** of subsets of X is a Boolean algebra $\mathfrak{A} \subset 2^X$ such that for any countable family $A_1, ..., A_n, ... \in \mathfrak{A}$ the union $\bigcup_{i=1}^\infty A_i$ is also an element of $\mathfrak{A}$.

REMARK: We define the operation of addition on the set $\mathbb{R} \cup \{\infty\}$ in such a way that $x + \infty = \infty$ and $\infty + \infty = \infty$. On finite numbers the addition is defined as usually.

DEFINITION: A function $\mu: \mathfrak{A} \longrightarrow \mathbb{R} \cup \{\infty\}$ is called **finitely additive** if for all non-intersecting $A, B \in \mathfrak{U}$, $\mu(A \coprod B) = \mu(A) + \mu(B)$. The sign $\coprod$ denotes union of non-intersecting sets. μ is called **σ-additive** if $\mu(\coprod_{i=1}^\infty A_i) = \sum \mu(A_i)$ for any pairwise disjoint countable family of subsets $A_i \in \mathfrak{A}$.

DEFINITION: **A measure** in a σ-algebra $\mathfrak{A} \subset 2^X$ is a σ-additive function $\mu: \mathfrak{A} \longrightarrow \mathbb{R} \cup \{\infty\}$.

EXAMPLE: Let X be a topological space. The **Borel σ-algebra** is a smallest σ-algebra $\mathfrak{A} \subset 2^X$ containing all open subsets. **Borel measure** is a measure on Borel σ-algebra.

Lebesgue measure

DEFINITION: **Pseudometric** on X is a function $d: X \times X \longrightarrow \mathbb{R}^{\geqslant 0}$ which is symmetric and satisfies the triangle inequality and $d(x,x) = 0$ for all $x \in X$. In other words, pseudometric is a metric which can take 0 on distinct points.

EXERCISE: Let $\mathfrak{A} \subset 2^X$ be a Boolean algebra with positive, additive function μ. Given $U, V \in 2^X$, denote by $U \triangle V$ their **symmetric difference**, that is, $U \triangle V := (U \cup V) \backslash (U \cap V)$. **Prove that the function $d_\mu(U,V) := \mu(U \triangle V)$ defines a pseudometric on $\mathfrak{A}$.**

DEFINITION: Let $\mathfrak{A} \subset 2^X$ be a Boolean algebra with positive, additive function μ. A set $U \subset X$ **has measure 0** if for each $\varepsilon > 0$, U can be covered by a union of $A_i \in \mathfrak{A}$, that is, $U \subset \bigcup_{i=1}^\infty A_i$, with $\sum_{i=0}^\infty \mu(A_i) < \varepsilon$.

REMARK: Consider a completion of $\mathfrak{A}$ with respect to the pseudometric d_μ. A limit of a Cauchy sequence $\{A_i\} \subset \mathfrak{A}$ can be realized as an element of 2^X; this realization is unique up to a set of measure 0. A set which can be obtained this way is called **a Lebesgue measurable set**. Extending μ to the metric completion of $\mathfrak{A}$ by continuity, we obtain **the Lebesgue measure** on the σ-algebra of Lebesgue measurable sets.

REMARK: This construction is also used **for constructing Borel measures.**

Measurable maps and measurable functions

DEFINITION: Let X, Y be sets equipped with σ-algebras $\mathfrak{A} \subset 2^X$ and $\mathfrak{B} \subset 2^Y$. We say that a map $f: X \longrightarrow Y$ is **compatible with the σ-algebra**, or **measurable**, if $f^{-1}(B) \in \mathfrak{A}$ for all $B \in \mathfrak{B}$.

REMARK: This is similar to the definition of continuity. In fact, **any continuous map of topological spaces is compatible with Borel σ-algebras.**

DEFINITION: Let X be a space with σ-algebra $\mathfrak{A} \subset 2^X$. A function $f: X \longrightarrow \mathbb{R}$ is called **measurable** if f is compatible with the Borel σ-algebra on $\mathbb{R}$, that is, if the preimage of any Borel set $A \subset \mathbb{R}$ belongs to $\mathfrak{A}$.

DEFINITION: Let X, Y be sets equipped with σ-algebras $\mathfrak{A} \subset 2^X$ and $\mathfrak{B} \subset 2^Y$, $f: X \longrightarrow Y$ a measurable map. Let μ be a measure on X. Consider the function $f_* \mu$ mapping $B \in \mathfrak{B}$ to $\mu(f^{-1}(B))$.

EXERCISE: Prove that $f_* \mu$ is a measure on Y.

DEFINITION: The measure $f_* \mu$ is called **the pushforward measure**, or **pushforward** of μ.

Integral

DEFINITION: Let $f : X \longrightarrow \mathbb{R}$ be a measurable function on a measured space (X, μ). We define **integral** $\int_X f\mu$ as an integral of the Borel measure in $\mathbb{R}$,

$$\int_X f\mu := \int_{\mathbb{R}} f_* \mu.$$

Of course, this definition assumes **we already know how to integrate Borel measurable functions on $\mathbb{R}$.**

Spaces with measure: examples

DEFINITION: Lebesgue measure on $\mathbb{R}^n$ is defined starting from the algebra $\mathfrak{A}$, generated by parallelepipeds with sides parallel to coordinate lines. The measure μ on $\mathfrak{A}$ takes a parallelepiped with sides $a_1, a_2, ..., a_n$ to $a_1 a_2 ... a_n$. The completion of this algebra with respect to μ is called **the algebra of Lebesgue measurable sets**. It contains all Borel sets.

DEFINITION: Let M be an oriented manifold, and Φ a positive volume form. For each coordinate patch $U_i \subset \mathbb{R}^n$, and a compact subset $K \subset U_i$, write Φ restricts to U_i as $\alpha dx_1 \wedge dx_2 \wedge ... dx_n$, with $\alpha \in C^\infty U_i$ a positive function. Let $\mu(K) := \int_K \alpha d\operatorname{Vol}$, where $\int_K \alpha dK$ is defined as above, and dK the Lebesgue measure on K. This is called **the Lebesgue measure on a manifold M associated with the volume form Φ**.

Spaces with measure: more examples

DEFINITION: Let P be a finite set, and $P^{\mathbb{Z}}$ the product of $\mathbb{Z}$ copies of P, and $\pi_i: \ P^{\mathbb{Z}} \longrightarrow P$ projection to the p-th component. Fix distinct numbers $i_1, ..., i_n \in \mathbb{Z}$ and let $K_1, ..., K_n \subset P$ be subsets. **Cylindrical set** is an intersection

$$C := \bigcup_{k=i_1,...,i_n} \pi_{i_k}^{-1}(K_k) \subset P^{\mathbb{Z}}.$$

Tychonoff topology, or **product topology** on $P^{\mathbb{Z}}$ is topology with the base consisting of all cylindrical sets. **Bernoulli measure** on $P^{\mathbb{Z}}$ is a measure μ such that $\mu(C) := \frac{\prod_{i=1}^n |K_i|}{|P|^n}$.

Bernoulli measure can be understood probabilistically as follows: we throw a dice with $|P|$ sides, randomly with equal probability chosing one of its sides, and look at the probability that i_k-th throw would land in the set $K_k \subset P$.

Categories

DEFINITION: A category $\mathcal{C}$ is a collection of data called "objects" and "morphisms between objects" which satisfies the axioms below.

DATA.

Objects: A class $\mathcal{O}\text{b}(\mathcal{C})$ of **objects** of $\mathcal{C}$.

Morphisms: For each $X, Y \in \mathcal{O}\text{b}(\mathcal{C})$, one has a set $\mathcal{M}\text{or}(X, Y)$ of **morphisms from X to Y.**

Composition of morphisms: For each $\varphi \in \mathcal{M}\text{or}(X, Y), \psi \in \mathcal{M}\text{or}(Y, Z)$ there exists **the composition** $\varphi \circ \psi \in \mathcal{M}\text{or}(X, Z)$

Identity morphism: For each $A \in \mathcal{O}\text{b}(\mathcal{C})$ there exists a morphism $\text{Id}_A \in \mathcal{M}\text{or}(A, A)$.

AXIOMS.

Associativity of composition: $\varphi_1 \circ (\varphi_2 \circ \varphi_3) = (\varphi_1 \circ \varphi_2) \circ \varphi_3$.

Properties of identity morphism: For each $\varphi \in \mathcal{M}\text{or}(X, Y)$, one has $\text{Id}_x \circ \varphi = \varphi = \varphi \circ \text{Id}_Y$

Categories (2)

DEFINITION: Let $X, Y \in \mathcal{O}b(\mathcal{C})$ – objects of $\mathcal{C}$. A morphism $\varphi \in \mathcal{M}or(X, Y)$ is called **an isomorphism** if there exists $\psi \in \mathcal{M}or(Y, X)$ such that $\varphi \circ \psi = \operatorname{Id}_X$ and $\psi \circ \varphi = \operatorname{Id}_Y$. In this case, the objects X and Y are called **isomorphic**.

Examples of categories:
Category of sets: its morphisms are arbitrary maps.
Category of vector spaces: its morphisms are linear maps.
Categories of rings, groups, fields: morphisms are homomorphisms.
Category of topological spaces: morphisms are continuous maps.
Category of smooth manifolds: morphisms are smooth maps.

Category of spaces with measure

DEFINITION: Let $\mathcal{C}$ be the **category of spaces with measure**, or **measured spaces**, where $\mathcal{O}\mathrm{b}(\mathcal{C})$ – spaces (X, μ_X) with measure, and $\mathcal{M}\mathrm{or}((X, \mu_X), (Y, \mu_Y))$ the set of all measurable maps $f: X \longrightarrow Y$ such that $f_* \mu_X = \mu_Y$.

REMARK: Isomorphism of spaces with measure is a **bijection which preserves the σ-algebra and the measure.**

OBSERVATION: Category of spaces with measure **is not very interesting.** Indeed, pretty much all measured spaces are isomorphic.

EXERCISE: Prove that **unit cubes of any given dimension are isomorphic as measured spaces.** Prove that **a unit cube is isomorphic to a Bernoulli space as a space with measure.**

Category of spaces with measure: exercises

Spaces with measure are very similar to the sets.

EXERCISE: (**"Cantor-Schröder-Bernstein theorem for measured spaces".**) Let X, Y spaces with measure, and $X_0 \subset X$, $Y_0 \subset Y$ measured subsets. Suppose that X_0 is isomorphic to Y and Y_0 is isomorphic to X as a space with measure. **Prove that X is isomorphic to Y.**

EXERCISE: Let C be a cube and $x \in C$ a point. **Prove that $C \backslash x$ is isomorphic to C as a space with measure.**

EXERCISE: Let C be a cube and $R \subset C$ a countable set. **Prove that $C \backslash R$ is isomorphic to C as a space with measure.**

Category of spaces with measure: more exercises

EXERCISE: Let $B = \{0,1\}^{\mathbb{Z}^{\geqslant 0}}$ be the set of all sequences of numbers $a_i \in \{0,1\}$ with Bernoulli measure, and $B_0 \subset B$ the set of all sequences not ending with an infinite string of "1". **Prove that B_0 is isomorphic, as a measured space to an interval $[0,1] \subset \mathbb{R}$.**

EXERCISE: Let $B = \{0,1\}^{\mathbb{Z}^{\geqslant 0}}$ be the set of all sequences of numbers $a_i \in \{0,1\}$ with Bernoulli measure. **Define the natural measure on the product of measured spaces**, and **prove that B is isomorphic to B^n as a space with measure for any $n > 0$.**

EXERCISE: Prove that **unit cubes of any given dimension are isomorphic as measured spaces.**