Teoria Ergódica Diferenciável

lecture 2: Poincare recurrence theorem

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σ-algebras and measures (reminder)

DEFINITION: Let M be a set **A σ-algebra** of subsets of X is a Boolean algebra $\mathfrak{A} \subset 2^X$ such that for any countable family $A_1, ..., A_n, ... \in \mathfrak{A}$ the union $\bigcup_{i=1}^\infty A_i$ is also an element of $\mathfrak{A}$.

REMARK: We define the operation of addition on the set $\mathbb{R} \cup \{\infty\}$ in such a way that $x + \infty = \infty$ and $\infty + \infty = \infty$. On finite numbers the addition is defined as usually.

DEFINITION: A function $\mu : \mathfrak{A} \longrightarrow \mathbb{R} \cup \{\infty\}$ is called **finitely additive** if for all non-intersecting $A, B \in \mathfrak{U}$, $\mu(A \coprod B) = \mu(A) + \mu(B)$. The sign $\coprod$ denotes union of non-intersecting sets. μ is called **σ-additive** if $\mu(\coprod_{i=1}^\infty A_i) = \sum \mu(A_i)$ for any pairwise disjoint countable family of subsets $A_i \in \mathfrak{A}$.

DEFINITION: **A measure** in a σ-algebra $\mathfrak{A} \subset 2^X$ is a σ-additive function $\mu : \mathfrak{A} \longrightarrow \mathbb{R} \cup \{\infty\}$.

EXAMPLE: Let X be a topological space. The **Borel σ-algebra** is a smallest σ-algebra $\mathfrak{A} \subset 2^X$ containing all open subsets. **Borel measure** is a measure on Borel σ-algebra.

Measurable maps and measurable functions (reminder)

DEFINITION: Let X, Y be sets equipped with σ-algebras $\mathfrak{A} \subset 2^X$ and $\mathfrak{B} \subset 2^Y$. We say that a map $f: X \longrightarrow Y$ is **compatible with the σ-algebra**, or **measurable**, if $f^{-1}(B) \in \mathfrak{A}$ for all $B \in \mathfrak{B}$.

REMARK: This is similar to the definition of continuity. In fact, **any continuous map of topological spaces is compatible with Borel σ-algebras.**

DEFINITION: Let X be a space with σ-algebra $\mathfrak{A} \subset 2^X$. A function $f: X \longrightarrow \mathbb{R}$ is called **measurable** if f is compatible with the Borel σ-algebra on $\mathbb{R}$, that is, if the preimage of any Borel set $A \subset \mathbb{R}$ belongs to $\mathfrak{A}$.

DEFINITION: Let X, Y be sets equipped with σ-algebras $\mathfrak{A} \subset 2^X$ and $\mathfrak{B} \subset 2^Y$, $f: X \longrightarrow Y$ a measurable map. Let μ be a measure on X. Consider the function $f_*\mu$ mapping $B \in \mathfrak{B}$ to $\mu(f^{-1}(B))$.

EXERCISE: Prove that $f_*\mu$ is a measure on Y.

DEFINITION: The measure $f_*\mu$ is called **the pushforward measure**, or **pushforward** of μ.

Category of spaces with measure (reminder)

DEFINITION: We define the **category of spaces with measure**, or **measured spaces**. Its objects are spaces (X, μ_X) with measure, and morphisms are measurable maps $f: X \longrightarrow Y$ such that $f_* \mu_X = \mu_Y$.

REMARK: Isomorphism of spaces with measure is a **bijection which preserves the σ-algebra and the measure.**

OBSERVATION: Category of spaces with measure **is not very interesting.** Indeed, pretty much all measured spaces are isomorphic.

EXERCISE: Prove that **unit cubes of any given dimension are isomorphic as measured spaces.** Prove that **a unit cube is isomorphic to a Bernoulli space as a space with measure.**

Poicaré recurrence theorem

DEFINITION: A measure μ on M is called **probabilistic** if $\mu(M) = 1$. A measurable subset $X \subset M$ is called **full measure subset** if $\mu(M \backslash X) = 0$.

DEFINITION: Let M be a topological space, and $\varphi : M \longrightarrow M$ a continuous map. **Recurrence set** of π is a set of all $x \in M$ such that for some unbounded sequence $\{m_i\}$ of natural numbers, one has $\lim_i \varphi^{m_i}(x) = x$.

THEOREM: (Poincaré recurrence theorem)
Let M be a second-countable metrisable topological space, μ a probabilistic Borel measure, and $\varphi : M \longrightarrow M$ a homeomorphism preserving measure. **Then the recurrence set R of φ has full measure.**

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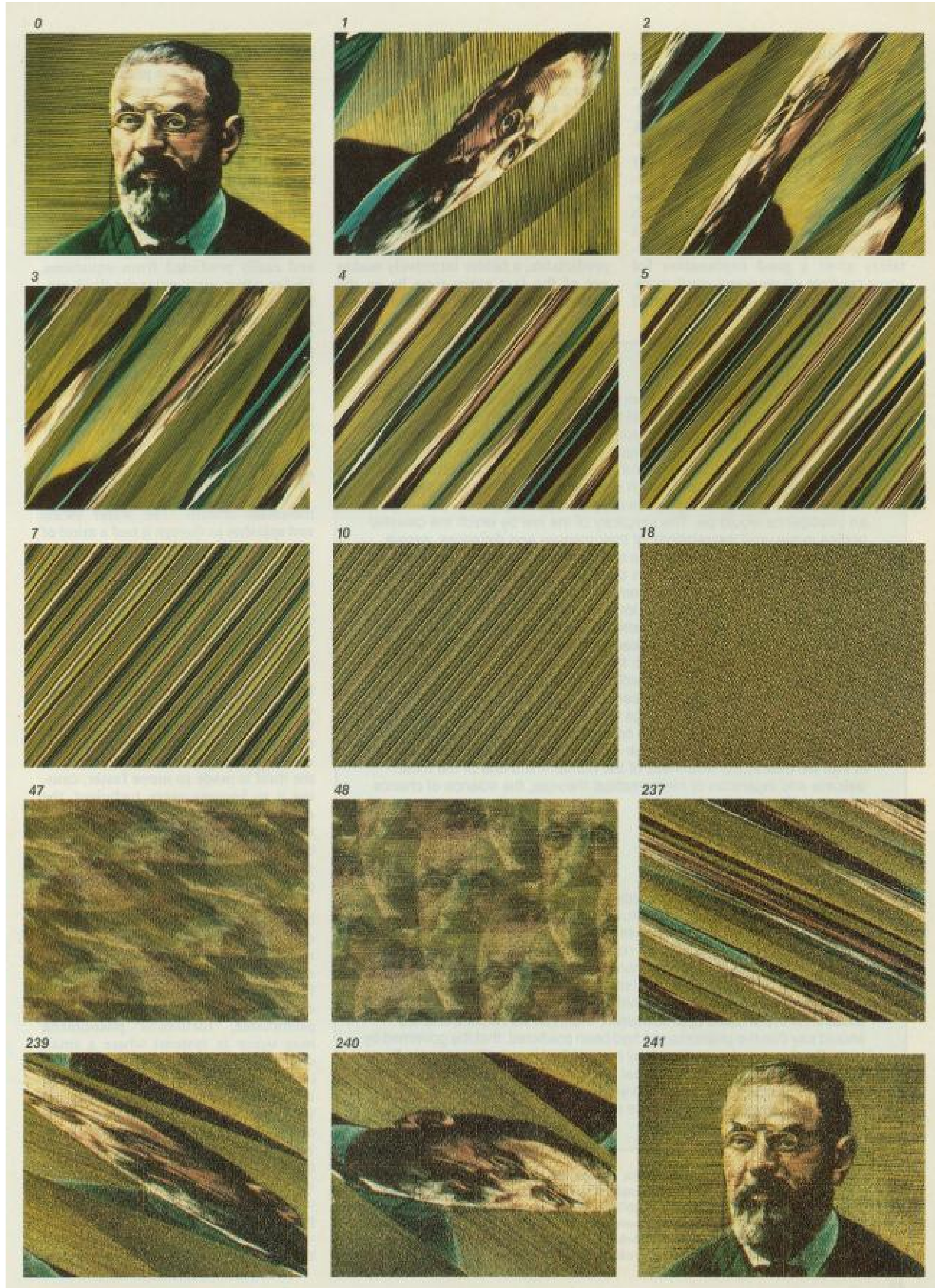
Proof. Step 1: Fix a metric on M, and met $B_\varepsilon(x)$ denote an ε-ball centered in x Define an **ε-recurrence set** R_ε as

$$R_\varepsilon := \{x \in M \mid B_\varepsilon(x) \cap \{\varphi(x), \varphi^2(x), \varphi^3(x), ...\} \neq 0\}$$

Then $R = \bigcap_\varepsilon R_\varepsilon$ **(prove it)**. To prove that R has full measure, **it would suffice to show that each R_ε has full measure.**

Step 2: Recall that **diameter** of a metric space B is $\operatorname{diam}(B) := \sup_{x,y\in B} d(x,y)$. Let $A_\varepsilon := M \backslash R_\varepsilon$. Suppose that A_ε has positive measure, and let $B \subset A_\varepsilon$ be a subset of positive measure and diameter ε. Since $\bigcup_i \varphi^i(B)$ has finite measure, for some $i \neq j$, **the sets $\varphi^i(B)$ and $\varphi^j(B)$ have non-trivial intersection.**

Step 3: Let $i > j$. Since $\varphi^i(B) \cap \varphi^j(B) \neq \emptyset$, there exists $x \in \varphi^{i-j}(B) \cap B$. Then $d(x, \varphi^{i-j}(x)) < \operatorname{diam}(B) \leqslant \varepsilon$, which implies that $x \notin A_\varepsilon$, **giving a contradiction.** ∎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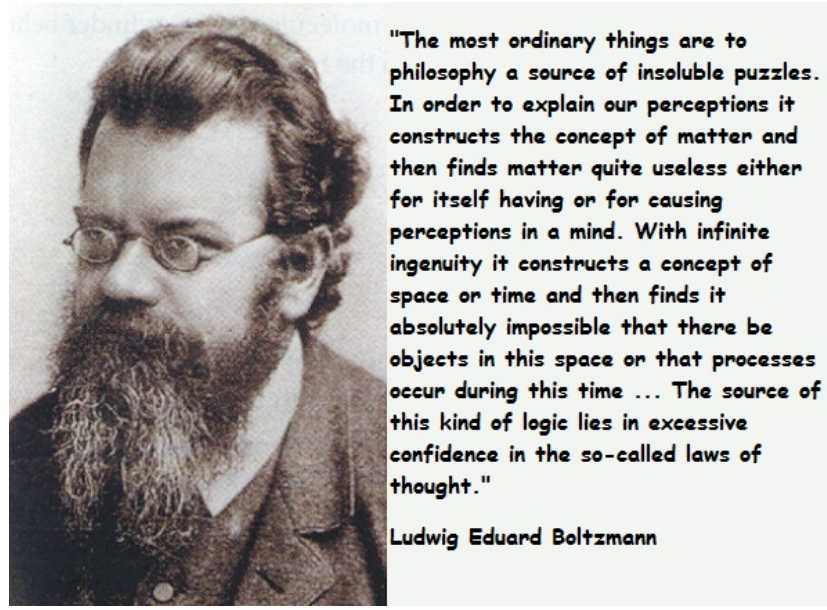
Heat death of the universe!

Jules Henri Poincaré
(1854 - 1912)



Ludwig Eduard Boltzmann
(1844 - 1906)

...I do not know if it has been remarked that the English kinetic theories can extricate themselves from this contradiction. The world, according to them, tends at first toward a state where it remains for a long time without apparent change; and this is consistent with experience; but it does not remain that way forever, if the theorem cited above is not violated; it merely stays there for an enormously long time, a time which is longer the more numerous are the molecules. This state will not be the final death of the universe, but a sort of slumber, from which it will awake after millions of millions of centuries. According to this theory, to see heat pass from a cold body to a warm one, it will not be necessary to have the acute vision, the intelligence, and dexterity of Maxwell's demon; it will suffice to have a little patience.

H. Poincare (1893) Le mécanisme et l'expérience.

Revue de Metaphysique et de Morale, 4, 534.

Heat death!

...One has the choice between two kinds of pictures. One can assume that the entire universe finds itself at present in a very improbable state. However, one may suppose that the aeons during which this improbable state lasts, and the distance from here to Sirius, are minute compared to the age and size of the universe. There must then be in the universe, which is in thermal equilibrium as a whole and therefore dead, here and there relatively small regions of the size of our galaxy (which we call worlds), which during the relatively short time of aeons deviate significantly from thermal equilibrium. Among these worlds the state probability increases as often as it decreases. For the universe as a whole the two directions of time are indistinguishable, just as in space there is no up and down. However, just as at a certain place on the earth we can call "down" the direction toward the centre of the earth, so a living being that finds itself in such a world at a certain period of time can define the time direction as going from less probable to more probable states (the former will be the "past", the latter the "future") and by virtue of this definition he will find that this small region, isolated from the rest of the universe, is "initially" always in an improbable state. This viewpoint seems to me the only way in which one can understand the validity of the Second Law and the heat death of each individual world, without invoking an unidirectional change of the entire universe from a definite initial state to final state...

L. Boltzmann (1897). Zu Hrn. Zermelo Abhandlung fiber die mechanische Erklarungen irreversible!' Vorgange. Wiedemann's Annalen, 60, 392-8.

Boltzmann's Ergodic Hypothesis: For large systems of interacting particles in equilibrium time averages are close to the ensemble, or equilibrium average.

er·god·ic

early 20th century: from German *ergoden*, from Greek *ergon* 'work' + *hodos* 'way' + -ic.

Earliest Known Uses of Some of the Words of Mathematics:

`http://jeff560.tripod.com/mathword.html`

ERGODIC. Ludwig Boltzmann (1844-1906) coined the term Ergode (from the Greek words for work + way) for what Gibbs later called a "micro-canonical ensemble"; Ergode appears in the 1884 article in Wien. Ber. 90, 231. Later P. & T. Ehrenfest (1911) "Begriffiche Grundlagen der statistischen Auffassung in der Mechanik" (Encyklopädie der mathematischen Wissenschaften, vol. 4, Part 32) discussed "ergodische mechanischer Systeme" the existence of which they saw as underlying the gas theory of Boltzmann and Maxwell. (Based on a note on p. 297 of Lectures on Gas Theory, S. G. Brush's translation of Boltzmann's Vorlesungen über Gastheorie.)

Ergodic measures

REMARK: Let M, μ be a space with measure. We say that **"property P holds for almost all $x \in M$"** when property P holds for all $x \in M$ outside of a measure 0 subset.

DEFINITION: Let Γ be a group acting on a measured space $(M.\mu)$ and preserving its σ-algebra. We say that the Γ-action is **ergodic** if for each Γ-invariant, measurable set $U \subset M$, either $\mu(U) = 0$ or $\mu(M\backslash U) = 0$. In this case μ is called **an ergodic measure**.

THEOREM: Let M be a second countable topological space, and μ a Borel measure on M. Let Γ be a group acting on M by homeomorphisms. Suppose that any non-empty open subset of M has positive measure, and action of Γ is ergodic. Then **for almost all $x \in M$, the orbit $\Gamma \cdot x$ is dense in M.**

Proof. Step 1: Let U_i be a countable base of topology on M. The orbit $\Gamma \cdot x$ is dense in M if $(\Gamma \cdot x) \cap U_i \neq 0$ for all i. This is equivalent to $x \in \Gamma \cdot U_i$. Therefore, **the set of all x with dense orbits is $\bigcap_i (\Gamma \cdot U_i)$.**

Step 2: Since $\Gamma \cdot U_i$ is Γ-invariant and has positive measure, it has full measure because of ergodicity. **Then $\bigcap_i (\Gamma \cdot U_i)$ is an intersection of sets which have full measure.** ■

Ergodic measures and integrable functions

Rule of a thumb: If your group action preserves measure and almost all its orbits are dense, it is most likely ergodic. **Not always!**

THEOREM: Let (M, μ) be a space with finite measure, and Γ a group acting on M and preserving the measure. Then the following are equivalent.

(a) The action of Γ is ergodic.

(b) For each integrable, Γ-invariant function $f: M \longrightarrow \mathbb{R}$, f is constant almost everywhere.

Proof: To obtain (a) from (b), take the characteristic function χ_U of a Γ-invariant set $U \subset M$. Then it is constant almost everywhere, hence U is of full measure (in this case $\chi_U = 1$ almost everywhere) or measure zero, in later case $\chi_U = 0$ almost everywhere.

To obtain (b) from (a), let c be the average value of f on M, and let $M_\varepsilon^+ := f^{-1}([c+\varepsilon, \infty[)$ and $M_\varepsilon^- := f^{-1}(]-\infty, c+\varepsilon])$. Both sets are Γ-invariant and not of full measure, hence they have measure zero. This means that for all $\varepsilon > 0$, $c - \varepsilon < f(x) < c + \varepsilon$ for almost all x. ■

Exercises (for discussion in class)

EXERCISE: Let α be an irrational number, and $\varphi_\alpha : S^1 \longrightarrow S^1$ be a rotation by $\pi\alpha$. **Prove that φ_α has dense orbits.**

EXERCISE: **Prove that φ_α is ergodic.**

DEFINITION: **(reminder)** Let P be a finite set, and $P^{\mathbb{Z}}$ the product of $\mathbb{Z}$ copies of P, and $\pi_i : P^{\mathbb{Z}} \longrightarrow P$ projection to the p-th component. Fix dictinct numbers $i_1, ..., i_n \in \mathbb{Z}$ and let $K_1, ..., K_n \subset P$ be subsets. **Cylindrical set** is an intersection

$$C := \bigcup_{k=i_1,...,i_n} \pi_{i_k}^{-1}(K_k) \subset P^{\mathbb{Z}}.$$

Tychonoff topology, or **product topology** on $P^{\mathbb{Z}}$ is topology with the base consisting of all cylindrical sets. **Bernoulli measure** on $P^{\mathbb{Z}}$ is a measure μ such that $\mu(C) := \frac{\prod_{i=1}^n |K_i|}{|P|^n}$.

DEFINITION: **Bernoulli shift** maps a sequence $a_{-n}, a_{-n+1}, ..., a_0, a_1, ...$ to the sequence $b_{-n}, b_{-n+1}, ..., b_0, b_1, ...,$ $b_i = a_{i-1}$.

EXERCISE: **Find a dense orbit for the Bernoulli shift.**

EXERCISE: **Prove that the Bernoulli shift is ergodic.**