Teoria Ergódica Diferenciável

lecture 4: Weak-$*$ topology on measures

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σ-algebras and measures (reminder)

DEFINITION: Let M be a set **A σ-algebra** of subsets of X is a Boolean algebra $\mathfrak{A} \subset 2^X$ such that for any countable family $A_1, ..., A_n, ... \in \mathfrak{A}$ the union $\bigcup_{i=1}^{\infty} A_i$ is also an element of $\mathfrak{A}$.

REMARK: We define the operation of addition on the set $\mathbb{R} \cup \{\infty\}$ in such a way that $x + \infty = \infty$ and $\infty + \infty = \infty$. On finite numbers the addition is defined as usually.

DEFINITION: A function $\mu : \mathfrak{A} \longrightarrow \mathbb{R} \cup \{\infty\}$ is called **finitely additive** if for all non-intersecting $A, B \in \mathfrak{U}$, $\mu(A \coprod B) = \mu(A) + \mu(B)$. The sign $\coprod$ denotes union of non-intersecting sets. μ is called **σ-additive** if $\mu(\coprod_{i=1}^{\infty} A_i) = \sum \mu(A_i)$ for any pairwise disjoint countable family of subsets $A_i \in \mathfrak{A}$.

DEFINITION: **A measure** in a σ-algebra $\mathfrak{A} \subset 2^X$ is a σ-additive function $\mu : \mathfrak{A} \longrightarrow \mathbb{R} \cup \{\infty\}$.

EXAMPLE: Let X be a topological space. The **Borel σ-algebra** is a smallest σ-algebra $\mathfrak{A} \subset 2^X$ containing all open subsets. **Borel measure** is a measure on Borel σ-algebra.

Measurable maps and measurable functions (reminder)

DEFINITION: Let X, Y be sets equipped with σ-algebras $\mathfrak{A} \subset 2^X$ and $\mathfrak{B} \subset 2^Y$. We say that a map $f: X \longrightarrow Y$ is **compatible with the σ-algebra**, or **measurable**, if $f^{-1}(B) \in \mathfrak{A}$ for all $B \in \mathfrak{B}$.

REMARK: This is similar to the definition of continuity. In fact, **any continuous map of topological spaces is compatible with Borel σ-algebras.**

DEFINITION: Let X be a space with σ-algebra $\mathfrak{A} \subset 2^X$. A function $f: X \longrightarrow \mathbb{R}$ is called **measurable** if f is compatible with the Borel σ-algebra on $\mathbb{R}$, that is, if the preimage of any Borel set $A \subset \mathbb{R}$ belongs to $\mathfrak{A}$.

DEFINITION: Let X, Y be sets equipped with σ-algebras $\mathfrak{A} \subset 2^X$ and $\mathfrak{B} \subset 2^Y$, $f: X \longrightarrow Y$ a measurable map. Let μ be a measure on X. Consider the function $f_*\mu$ mapping $B \in \mathfrak{B}$ to $\mu(f^{-1}(B))$.

EXERCISE: **Prove that $f_*\mu$ is a measure on Y.**

DEFINITION: The measure $f_*\mu$ is called **the pushforward measure**, or **pushforward** of μ.

Ergodic measures (reminder)

DEFINITION: Let Γ be a group acting on a measured space $(M.\mu)$ and preserving its σ-algebra. We say that the Γ-action is **ergodic** if for each Γ-invariant, measurable set $U \subset M$, either $\mu(U) = 0$ or $\mu(M \backslash U) = 0$. In this case μ is called **an ergodic measure**.

THEOREM: Let M be a second countable topological space, and μ a Borel measure on M. Let Γ be a group acting on M by homeomorphisms. Suppose that any non-empty open subset of M has positive measure, and action of Γ is ergodic. Then **for almost all $x \in M$, the orbit $\Gamma \cdot x$ is dense in M.**

THEOREM: Let (M, μ) be a space with finite measure, and Γ a group acting on M and preserving the measure. Then the following are equivalent.

(a) The action of Γ is ergodic.

(b) For each integrable, Γ-invariant function $f : M \longrightarrow \mathbb{R}$, f is constant almost everywhere.

(c) For each square integrable, Γ-invariant function $f : M \longrightarrow \mathbb{R}$, f is constant almost everywhere.

Radon-Nikodym theorem

DEFINITION: Let S be a space equipped with a σ-algebra, and μ, ν two measures on this σ-algebra. We say that ν is **absolutely continuous** with respect to μ if for each measurable set A, $\mu(A) = 0$ implies $\nu(A) = 0$. This relation is denoted $\nu \ll \mu$; clearly, it defines a partial order on measures.

EXERCISE: Find an example of a Borel measure on $\mathbb{R}^n$ which is **not absolutely continuous with respect to the usual Lebesgue measure.**

EXERCISE: Find an infininite family $\mathfrak{M}$ of measures on $\mathbb{R}^n$ such that **each measure $\mu \in \mathfrak{M}$ is not absolutely continuous with respect to each other $\mu' \in \mathfrak{M}$.**

EXERCISE: Let μ be a measure on a space M with σ-algebra, and $f : M \longrightarrow \mathbb{R}^{\geqslant 0}$ an integrable function. Define a measure $f\mu$ by $A \longrightarrow \int_A f\mu$. **Prove that $f\mu \ll \mu$.**

THEOREM: (Radon-Nikodym) Let μ, ν be two measures on a space S with a σ-algebra, satisfying $\mu(S) < \infty$, $\nu(S) < \infty$ and $\nu \ll \mu$. **Then there exists an integrable function $f : S \longrightarrow \mathbb{R}^{\geqslant 0}$ such that $\nu = f\mu$.**

Proof: I will distribute it at certain point. ∎

Convex cones and extremal rays

DEFINITION: Let V be a vector space over $\mathbb{R}$, and $K \subset V$ a subset. We say that K is **convex** if for all $x, y \in K$, the interval $\alpha x + (1-\alpha)y$, $\alpha \in [0,1]$ lies in K. We say that K is **a convex cone** if it is convex and for all $\lambda > 0$, the homothety map $x \longrightarrow \lambda x$ preserves K.

EXAMPLE: Let M be a space equipped with a σ-algebra $\mathfrak{A} \subset 2^M$, and V the space formally generated by all $X \in \mathfrak{A}$. Denote by $\mathcal{S}$ subspace in V^* generated by all finite measures. This space is called **the space of finite signed measures**. **The measures constitute a convex cone in $\mathcal{S}$.**

DEFINITION: **Extreme point** of a convex set K is a point $x \in K$ such that for any $a, b \in K$ and any $t \in [0,1]$, $ta + (1-t)b = x$ implies $a = b = x$. **Extremal ray** of a convex cone K is a non-zero vector x such that for any $a, b \in K$ and $t_1, t_2 > 0$, a decomposition $x = t_1 a + t_2 b$ implies that a, b are proportional to x.

DEFINITION: **Convex hull** of a set $X \subset V$ is the smallest convex set containing X.

EXAMPLE: Let V be a vector space, and $x_1, ..., x_n, ...$ linearly independent vectors. **Simplex** is the convex hull of $\{x_i\}$. Its extremal points are $\{x_i\}$ **(prove it)**.

Ergodic measures as extremal rays (1)

Lemma 1: Let (M, μ) be a measured space, and Γ a group which acts ergodically on M. Consider a measure ν on M which is Γ-invariant and satisfies $\nu \ll \mu$. **Then** $\nu = \mathrm{const} \cdot \mu$.

Proof: Radon-Nikodym gives $\nu = f\mu$. The function $f = \frac{\nu}{\mu}$ is Γ-invariant, because both ν and μ are Γ-invariant. Then $f = \mathrm{const}$ almost everywhere. ∎

Lemma 2: Let μ_1, μ_2 be measures, $t_1, t_2 \in \mathbb{R}^{>0}$, and $\mu := t_1\mu_1 + t_2\mu_2$. **Then** $\mu_1 \ll \mu$.

Proof: $\mu_1(U) \leqslant t_1^{-1}\mu(U)$, hence $\mu_1(U) = 0$ whenever $\mu(U) = 0$. ∎

Ergodic measures as extremal rays (2)

THEOREM: Let (M, μ) be a space equipped with a σ-algebra and a group Γ acting on M and preserving the σ-algebra, and $\mathcal{M}$ the cone of finite inivariant measures on M. Consider a finite, Γ-invariant measure on M. Then the following are equivalent.

(a) $\mu \in \mathcal{M}$ lies in the extremal ray of $\mathcal{M}$

(b) μ is ergodic.

(a) implies (b): Let U be an Γ-invariant measurable subset. Then $\mu = \mu|_U + \mu\Big|_{M\backslash U}$, and one of these two measures must vanish, because μ is extremal.

(b) implies (a): Let $\mu = \mu_1 + \mu_2$ be a decomposition of the measure μ onto a sum of two invariant measures. Then $\mu \gg \mu_1$ and $\mu \gg \mu_2$ (Lemma 2), hence μ is proportional to μ_1 and μ_2 (Lemma 1). ∎

REMARK: A probability measure μ **lies on an extremal ray if and only if it is extreme as a point in the convex set of all probability measures (prove it).**

Existence of ergodic measures: strategy

To prove existence of ergodic measures, we shall use the following strategy:

1. Define topology on the space $\mathcal{M}$ of finite measures (**"measure topology"** or **"weak-$*$ topology"**) such that the space of probability measures is compact.

2. Prove Krein-Milman theorem

THEOREM: (Krein-Milman) Let $K \subset V$ be a compact, convex subset in a locally convex topological vector space. **Then K is the closure of the convex hull of the set of its extreme points.**

This theorem implies that any Γ-invariant finite measure is a limit of finite sums of ergodic measures.

EXERCISE: Find all ergodic measures on a cube with trivial group action and the standard measure.

Weak-$*$ topology

DEFINITION: Let M be a topological space, and $C^0_c(M)$ the space of continuous function with compact support. Any finite Borel measure μ defines a functional $C^0_c(M) \longrightarrow \mathbb{R}$ mapping f to $\int_M f\mu$. We say that a sequence $\{\mu_i\}$ of measures **converges in weak-$*$ topology** (or **in measure topology**) to μ if

$$\lim_i \int_M f\mu_i = \int_M f\mu$$

for all $f \in C^0_c(M)$. **The base of open sets of weak-$*$ topology** is given by $U_{f,]a,b[}$ where $]a,b[\subset \mathbb{R}$ is an interval, and $U_{f,]a,b[}$ is the set of all measures μ such that $a < \int_M f\mu < b$.

Tychonoff topology

DEFINITION: Let $\{X_\alpha\}$ be a family of topological spaces, parametrized by $\alpha \in \mathcal{I}$. **Product topology**, or **Tychonoff topology** on the product $\prod_\alpha X_\alpha$ is topology where the open sets are generated by unions and finite intersections of $\pi_a^{-1}(U)$, where $\pi_a : \prod_\alpha X_\alpha$ is a projection to the X_a-component, and $U \subset X_a$ is an open set.

REMARK: Tychonoff topology is also called **topology of pointwise convergence**, because the points of $\prod_\alpha X_\alpha$ can be considered as maps from the set of indices $\mathcal{I}$ to the corresponding X_α, and a sequence of such maps converges if and only if it converges for each $\alpha \in \mathcal{I}$.

REMARK: Consider a finite measure as an element in the product of $C^0_c(M)$ copies of $\mathbb{R}$, that is, as a continuous map from $C^0_c(M)$ to $\mathbb{R}$. **Then the weak-$*$ topology is induced by the Tychonoff topology on this product.**

Measures as functionals on $C_c^0(M)$

DEFINITION: Locally finite measure is a Borel measure which is finite on a certain base of open sets.

DEFINITION: Uniform topology on functions is induced by the metric $d(f,g) = \sup|f-g|$.

Theorem (*): Let M be a metrizable, locally compact topological vector space, and $C_c^0(M)^*$ the space of functionals continuous in uniform topology. **Then locally finite measures can be characterized as elements $\mu \in C_c^0(M)^*$ which are non-negative on all non-negative functions.**

Proof: Clearly, all measures give such functionals. Conversely, consider a functional $\mu \in C_c^0(M)^*$ which is non-negative on all non-negative functions. Given a closed set $K \subset M$, the characteristic function χ_K can be obtained as a monotonously decreasing limit of continuous functions f_i which are equal to 1 on K **(prove it).** Define $\mu(K) := \lim_i \mu(f_i)$; this limit is well defined because the sequence $\mu(f_i)$ is monotonous. This gives an additive Borel measure on M **(prove it).** ■

Space of measures and Tychonoff topology

REMARK: (**Tychonoff theorem**)
A product of any number of compact spaces is compact.

This theorem is hard and its proof is notoriously counter-intiutive. However, from Tychonoff the following theorem follows immediately.

THEOREM: Let M be a compact topological space, and $\mathcal{P}$ the space of probability measures on M equipped with the measure topology. **Then $\mathcal{P}$ is compact.**

Proof. Step 1: For any probability measure on M, and any $f \in C_c^0(M)$, one has $\min(f) \leqslant \int_M f\mu \leqslant \max(f)$. Therefore, μ can be considered as an element of the product $\prod_{f \in C_c^0(M)}[\min(f), \max(f)]$ of closed intervals indexed by $f \in C_c^0(M)$, and **Tychonoff topology on this product induces the weak-$*$ topology.**

Step 2: A closed subset of a compact set is again compact, hence **it suffices to show that all limit points of $\mathcal{P} \subset \prod_{f \in C_c^0(M)}[\min(f), \max(f)]$ are probability measures.** This is implied by Theorem (*). The limit measure satisfies $\mu(M) = 1$ because the constant function $f = 1$ has compact support, hence $\lim \int_M \mu_i = \int_M \mu$ whenever $\lim_i \mu_i = \mu$. ■

The space $C^0_c(M)$ is second countable (an exercise)

DEFINITION: Let $C \in \mathbb{R}^{>0}$. A function $f: M \longrightarrow \mathbb{R}$ is called **C-Lipschitz** if $|f(x) - f(y)| < Cd(x,y)$, and **Lipschitz** if it is C-Lipschitz for some $C > 0$.

EXERCISE: Let M be a second countable metrizable topological space. **Prove that the space of all Lipschitz maps with uniform topology has a countable dense subset.**

EXERCISE: Let M be a second countable metrizable topological space. **Prove that $C^0_c(M)$ has a countable dense subset.**

The space of Lipschitz functions is second countable

DEFINITION: An ε-**net** in a metric space M is a subset $Z \subset M$ such that any $m \in M$ lies in an ε-ball with center in Z.

REMARK: A metric space is compact **if and only if it has a finite ε-net for each $\varepsilon > 0$** **(prove it)**.

Claim 1: Let M be a compact metrizable topological space. **Then the space of C-Lipschitz functions has a countable dense subset.**

Proof. Step 1: Let Z be a finite ε/C-net in M_0. Then for any C-Lipschitz functions f, g, one has

$$\left| \sup_{m \in M} |f - g| - \sup_{z \in Z} |f - g| \right| < 2\varepsilon,$$

because for each $m \in M$ there exists $m' \in Z$ such that $d(m, m') < \varepsilon/C$, and then $|f(m) - f(m')| < C\varepsilon/C = \varepsilon$, giving $|f(m) - g(m)| < |f(m') - g(m')| + 2\varepsilon$.

The space of Lipschitz functions is second countable

Proof. Step 1: Let Z be a finite ε/C-net in M_0. Then for any C-Lipschitz functions f, g,

$$\left|\sup_{m\in M} |f-g| - \sup_{z\in Z} |f-g|\right| < 2\varepsilon.$$

Step 2: Let R_ε be the set of all functions on Z with values in $\mathbb{Q}$. For each $\varphi \in R_\varepsilon$ denote by U_φ an open set of all C-Lipschitz functions f satisfying $\max_{z\in Z} |f(z)-\varphi(z)| < \varepsilon$. Then for all $f, g \in U_\varphi$, one has $\max_{z\in Z} |f(z)-g(z)| < 2\varepsilon$, and by Step 1 this gives $\sup_{m\in M} |f-g| < 4\varepsilon$.

Step 3: The set of all such U_φ is countable; choosing a function f_φ in each non-empty U_φ, we use $\sup_{m\in M} |f-g| < 4\varepsilon$ to see that $\{f_\varphi\}$ is a countable 4ε-net in the space of C-Lipschitz functions. ■

COROLLARY: Let M be a compact metrizable topological space. **Then $C^0_c(M)$ has a countable dense subset.**

Proof: Using Claim 1, we see that it is sufficient to show that Lipschitz functions are dense in the set of all continuous functions; this follows from the Stone-Weierstrass theorem. ■

Tychonoff theorem for countable families

REMARK: Let $\{F_i\}$ be a countable, dense set in $C^0(M)$. Then any measure μ is determined by $\int_M F_i\mu$, and **weak-$*$ topology is topology of pointwise convergence on F_i**. This implies that **compactness of the space of measures is implied by the compactness of the product $\prod_{F_i}[\min(F_i), \max(F_i)]$, which is countable.**

THEOREM: (Countable Tychonoff theorem)
A countable product of metrizable compacts is compact.

Proof: Let $\{M_i\}$ be a countable family of metrizable compacts. We need to show that the space of sequences $\{a_i \in M_i\}$ with topology of pointwise convergence is compact. Take a sequence $\{a_i(j)\}$ of such sequences, and replace it by a subsequence $\{a'_i(j) \in M_i\}$ where $a_1(i)$ converges. Let $b_1 := \lim a'_i(1)$. Replace this sequence by a subsequence $\{a''_i(j) \in M_i\}$ where $a_2(i)$ converges. Put $b_2 = \lim_i a''_i(2)$ and so on. Then $\{b_i\}$ is a limit point of our original sequence $\{a_i(j)\}$. By Heine-Borel, compactness for second countable spaces is equivalent to sequential compactness, hence $\prod_i M_i$ is compact. ■