

Teoria Ergódica Diferenciável

lecture 5: Weak-* topology and Birkhoff Ergodic Theorem

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Weak-* topology (reminder)

DEFINITION: Let M be a topological space, and $C_c^0(M)$ the space of continuous function with compact support. Any finite Borel measure μ defines a functional $C_c^0(M) \rightarrow \mathbb{R}$ mapping f to $\int_M f \mu$. We say that a sequence $\{\mu_i\}$ of measures **converges in weak-* topology** (or **in measure topology**) to μ if

$$\lim_i \int_M f \mu_i = \int_M f \mu$$

for all $f \in C_c^0(M)$. **The base of open sets of weak-* topology** is given by $U_{f,]a,b[}$ where $]a,b[\subset \mathbb{R}$ is an interval, and $U_{f,]a,b[}$ is the set of all measures μ such that $a < \int_M f \mu < b$.

Tychonoff topology (reminder)

DEFINITION: Let $\{X_\alpha\}$ be a family of topological spaces, parametrized by $\alpha \in \mathcal{I}$. **Product topology**, or **Tychonoff topology** on the product $\prod_\alpha X_\alpha$ is topology where the open sets are generated by unions and finite intersections of $\pi_\alpha^{-1}(U)$, where $\pi_\alpha : \prod_\alpha X_\alpha$ is a projection to the X_α -component, and $U \subset X_\alpha$ is an open set.

REMARK: Tychonoff topology is also called **topology of pointwise convergence**, because the points of $\prod_\alpha X_\alpha$ can be considered as maps from the set of indices \mathcal{I} to the corresponding X_α , and a sequence of such maps converges if and only if it converges for each $\alpha \in \mathcal{I}$.

REMARK: Consider a finite measure as an element in the product of $C_c^0(M)$ copies of \mathbb{R} , that is, as a continuous map from $C_c^0(M)$ to \mathbb{R} . **Then the weak-* topology is induced by the Tychonoff topology on this product.**

Measures as functionals on $C_c^0(M)$

DEFINITION: Locally finite measure is a Borel measure which is finite on a certain base of open sets.

DEFINITION: Uniform topology on functions is induced by the metric $d(f, g) = \sup |f - g|$.

Theorem (*): Let M be a metrizable, locally compact topological vector space, and $C_c^0(M)^*$ the space of functionals continuous in uniform topology. **Then locally finite measures can be characterized as elements $\mu \in C_c^0(M)^*$ which are non-negative on all non-negative functions.**

Proof: Clearly, all measures give such functionals. Conversely, consider a functional $\mu \in C_c^0(M)^*$ which is non-negative on all non-negative functions. Given a closed set $K \subset M$, the characteristic function χ_K can be obtained as a monotonously decreasing limit of continuous functions f_i which are equal to 1 on K (**prove it**). Define $\mu(K) := \lim_i \mu(f_i)$; this limit is well defined because the sequence $\mu(f_i)$ is monotonous. This gives an additive Borel measure on M (**prove it**). ■

Space of measures and Tychonoff topology (reminder)

REMARK: (Tychonoff theorem)

A product of any number of compact spaces is compact.

This theorem is hard and its proof is notoriously counter-intuitive. However, from Tychonoff the following theorem follows immediately.

THEOREM: Let M be a compact topological space, and \mathcal{P} the space of probability measures on M equipped with the measure topology. **Then \mathcal{P} is compact.**

Proof. Step 1: For any probability measure on M , and any $f \in C_c^0(M)$, one has $\min(f) \leq \int_M f \mu \leq \max(f)$. Therefore, μ can be considered as an element of the product $\prod_{f \in C_c^0(M)} [\min(f), \max(f)]$ of closed intervals indexed by $f \in C_c^0(M)$, and **Tychonoff topology on this product induces the weak-* topology.**

Step 2: A closed subset of a compact set is again compact, hence **it suffices to show that all limit points of $\mathcal{P} \subset \prod_{f \in C_c^0(M)} [\min(f), \max(f)]$ are probability measures.** This is implied by Theorem (*). The limit measure satisfies $\mu(M) = 1$ because the constant function $f = 1$ has compact support, hence $\lim \int_M \mu_i = \int_M \mu$ whenever $\lim_i \mu_i = \mu$. ■

The space of Lipschitz functions is second countable

DEFINITION: An ε -net in a metric space M is a subset $Z \subset M$ such that any $m \in M$ lies in an ε -ball with center in Z .

REMARK: A metric space is compact **if and only if it has a finite ε -net for each $\varepsilon > 0$ (prove it).**

Claim 1: Let M be a compact metrizable topological space. **Then the space of C -Lipschitz functions has a countable dense subset.**

Proof. Step 1: Let Z be a finite ε/C -net in M_0 . Then for any C -Lipschitz functions f, g , one has

$$\left| \sup_{m \in M} |f - g| - \sup_{z \in Z} |f - g| \right| < 2\varepsilon,$$

because for each $m \in M$ there exists $m' \in Z$ such that $d(m, m') < \varepsilon/C$, and then $|f(m) - f(m')| < C\varepsilon/C = \varepsilon$, giving $|f(m) - g(m)| < |f(m') - g(m')| + 2\varepsilon$.

The space of Lipschitz functions is second countable (2)

Proof. Step 1: Let Z be a finite ε/C -net in M_0 . Then for any C -Lipschitz functions f, g ,

$$\left| \sup_{m \in M} |f - g| - \sup_{z \in Z} |f - g| \right| < 2\varepsilon.$$

Step 2: Let R_ε be the set of all functions on Z with values in \mathbb{Q} . For each $\varphi \in R_\varepsilon$ denote by U_φ an open set of all C -Lipschitz functions f satisfying $\max_{z \in Z} |f(z) - \varphi(z)| < \varepsilon$. Then for all $f, g \in U_\varphi$, one has $\max_{z \in Z} |f(z) - g(z)| < 2\varepsilon$, and by Step 1 this gives $\sup_{m \in M} |f - g| < 4\varepsilon$.

Step 3: The set of all such U_φ is countable; choosing a function f_φ in each non-empty U_φ , we use $\sup_{m \in M} |f - g| < 4\varepsilon$ to see that $\{f_\varphi\}$ is a countable 4ε -net in the space of C -Lipschitz functions. ■

COROLLARY: Let M be a compact metrizable topological space. **Then $C_c^0(M)$ has a countable dense subset.**

Proof: Using Claim 1, we see that it is sufficient to show that Lipschitz functions are dense in the set of all continuous functions; this follows from the Stone-Weierstrass theorem. ■

Tychonoff theorem for countable families

REMARK: Let $\{F_i\}$ be a countable, dense set in $C^0(M)$. Then any measure μ is determined by $\int_M F_i \mu$, and **weak-* topology is topology of pointwise convergence on F_i** . This implies that **compactness of the space of measures is implied by the compactness of the product $\prod_{F_i} [\min(F_i), \max(F_i)]$, which is countable.**

THEOREM: (Countable Tychonoff theorem)

A countable product of metrizable compacts is compact.

Proof: Let $\{M_i\}$ be a countable family of metrizable compacts. We need to show that the space of sequences $\{a_i \in M_i\}$ with topology of pointwise convergence is compact. Take a sequence $\{a_i(j)\}$ of such sequences, and replace it by a subsequence $\{a'_i(j) \in M_i\}$ where $a_1(i)$ converges. Let $b_1 := \lim a'_i(1)$. Replace this sequence by a subsequence $\{a''_i(j) \in M_i\}$ where $a_2(i)$ converges. Put $b_2 = \lim_i a''_i(2)$ and so on. Then $\{b_i\}$ is a limit point of our original sequence $\{a_i(j)\}$. By Heine-Borel, compactness for second countable spaces is equivalent to sequential compactness, hence $\prod_i M_i$ is compact. ■

Fréchet spaces

DEFINITION: A **seminorm** on a vector space V is a function $\nu : V \rightarrow \mathbb{R}^{\geq 0}$ satisfying

1. $\nu(\lambda x) = |\lambda|\nu(x)$ for each $\lambda \in \mathbb{R}$ and all $x \in V$
2. $\nu(x + y) \leq \nu(x) + \nu(y)$.

DEFINITION: We say that **topology on a vector space V is defined by a family of seminorms $\{\nu_\alpha\}$** if the base of this topology is given by the finite intersections of the sets

$$B_{\nu_\alpha, \varepsilon}(x) := \{y \in V \mid \nu_\alpha(x - y) < \varepsilon\}$$

("open balls with respect to the seminorm"). It is **complete** if each sequence $x_i \in V$ which is Cauchy with respect to each of the seminorms converges.

DEFINITION: A **Fréchet space** is a Hausdorff second countable topological vector space V with the topology defined by a countable family of seminorms, complete with respect to this family of seminorms.

Seminorms and weak-* topology

REMARK: Let M be a manifold and W be the subspace in functionals on $C_c^0(M)$ generated by all Borel measures ("the space of signed measures"). Recall that **the Hahn decomposition** is a decomposition of $\mu \in W$ as $\mu = \mu_+ - \mu_-$, where μ_+, μ_- are measures with non-intersecting support.

EXAMPLE: Then the weak-* topology is defined by a countable family of seminorms. Indeed, we can choose a dense, countable family of functions $f_i \in C_c^0(M)$, and define the seminorms ν_{f_i} on measures by $\nu_{f_i}(\mu) := \int_M f_i \mu$ extending it to W by $\nu_{f_i}(\mu) = \int_M f_i \mu_+ + \int_M f_i \mu_-$, where $\mu = \mu_+ - \mu_-$ is the Hahn decomposition.

EXERCISE: Prove that **the space W of signed measures with weak-* topology is complete.**

REMARK: This exercise is hard, but **for our purposes it is sufficient to replace W by its seminorm completion \overline{W} .** Since the space of finite measures is compact, it is also complete in \overline{W} .

Fréchet spaces: some examples

EXERCISE: Prove that the space of continuous functions on \mathbb{R}^n with topology of uniform convergence with compact support is Fréchet.

EXERCISE: Prove that the space of smooth functions on a compact manifold admits a structure of a Fréchet space.

Existence of invariant measures

Further on, we shall prove the following theorem

Theorem 1: Let $K \subset V$ be a compact, convex subset of a topological vector space with topology defined by a family of seminorms, and $A : V \rightarrow V$ a continuous linear map which preserves K . **Then there exists a point $z \in K$ such that $A(z) = z$.**

We shall prove this theorem in the next slide.

COROLLARY: Let M be a compact topological space and $f : M \rightarrow M$ a continuous map. **Then there exists an f -invariant probability measure on M .**

Proof: Take the compact space $K \subset W$ of all probability measures, and let $A : K \rightarrow K$ map μ to $f_*\mu$. Then A has a fixed point, as follows from Theorem 1. ■

Linear maps on convex compact sets

Theorem 1: Let $K \subset V$ be a compact, convex subset of a topological vector space with topology defined by a family of seminorms, and $A : V \rightarrow V$ a continuous linear map which preserves K . **Then there exists a point $z \in K$ such that $A(z) = z$.**

Proof: Consider the linear map $A_n(x) := \frac{1}{n} \sum_{i=0}^{n-1} A^i(x)$. Since it is an average of points in K , one has $A_n(x) \in K$. Let $z \in K$ be a limit point of the sequence $\{A_n(x)\}$ for some $x \in K$. Since

$$(1 - A)A_n(x) = \frac{(1 - A) \left(\sum_{i=0}^{n-1} A^i(x) \right)}{n} = \frac{1 - A^n(x)}{n},$$

for each seminorm ν_i on V one has

$$\nu(A(A_n(x)) - A_n(x)) < \frac{C}{n},$$

where

$$C := \sup_{x, y \in K} \nu(x - y).$$

By continuity of ν , this gives $\nu(A(z) - z) < \frac{C}{n}$ for each $n > 0$, hence $A(z) = z$.

■

Linear maps on convex compact sets: properties of the limit

Claim 1: Let $K \subset V$ be a compact, convex subset of a topological vector space with topology defined by a family of seminorms, and $A : V \rightarrow V$ a continuous linear map which preserves K . Consider the map $A_n(x) := \frac{1}{n} \sum_{i=0}^{n-1} A^i(x)$, and let $\text{Map}(K, K)$ be the space of maps from K to itself with the Tychonoff topology. Then $\{A_n\}$ has a subsequence converging to a linear map B from K to itself. Consider B as a linear map from the space $V' \subset V$ generated by K to itself. For two such limits B_1 and B_2 , the difference $E := B_1 - B_2$ satisfies $\text{im } E \subset V_0$, $\ker E \subset V_0$, where $V_0 = \ker(1 - A) \cap V'$.

Proof. Step 1: Consider the space $\text{Map}(K, K)$ of maps from K to itself with the product topology. By Tychonoff theorem, it is compact. The set of linear maps is closed in $\text{Map}(K, K)$ (**prove it**). Then **the sequence $\{A_n \in \text{Map}(K, K)\}$ has a limit point $B : K \rightarrow K$** which is a linear map on K . Then **B defines a linear (possibly discontinuous) endomorphism of V'** .

Step 2: Since $(1 - A)A_n(x) = \frac{1 - A^n}{n}$, one has $(1 - A)B = B(1 - A) = 0$. This implies that $\text{im } B \subset V_0$. Since $B|_{V_0} = A$, we also have $E|_{V_0} = 0$. ■

Birkhoff Ergodic Theorem

Lemma 1: Let $C > 0$ be a constant, ν a measure on S , and $K_{C,\nu}$ be the space of measures μ on S which satisfy $\mu(U) \leq C\nu(U)$ for all measurable sets. **Then $K_{C,\nu}$ is closed in weak-* topology.** **Proof:** $K_{C,\nu} = \bigcap_{f \in C_c^0(M)} K_f$, where $K_f = \{\text{measures } \mu \mid \int_S |f| \mu \leq C \int_S |f| \nu.\}$ ■

THEOREM: (Birkhoff Ergodic Theorem) Let $f : M \rightarrow M$ be a continuous map on a compact topological space, and μ a probability measure. Assume that $\mu = \Phi\nu$, where $f_*\nu = \nu$, and $|\Phi| < C$ is a bounded measurable function. **Then the sequence $\mu_n := \frac{1}{n} \sum_{i=0}^{n-1} (f_*)^i \mu$ converges to a probability measure.**

Proof. Step 1: The sequence $\mu_n := \frac{1}{n} \sum_{i=0}^{n-1} (f_*)^i \mu$ has a limit point μ' which is absolutely continuous with respect to ν by Lemma 1. Moreover, the function $\Psi := \frac{\mu'}{\nu}$ is bounded by the same constant C . Since $|\mu_n - f_*\mu_n| < \frac{|\mu_n| - |f_*\mu_n|}{n}$, **the limit function Ψ is f -invariant.**

Step 2: Consider the map $E : K \rightarrow V_0$ of Claim 1. Restricted to functions which we consider as signed measures, this map defines an f_* -invariant V_0 -valued functional $\tilde{\nu} : C_c^0(M) \rightarrow V_0$ which is absolutely continuous with respect to μ . Composing it with a linear functional, and applying Radon-Nikodym, we obtain an integrable f_* -invariant function $\Phi \in L^1(M)$. Then $\nu(\Phi) \neq 0$, because $\int_M \Phi^2 > 0$. This is impossible, because $E|_{V_0} = 0$. ■