Teoria Ergódica Diferenciável

lecture 6: Hopf argument

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Volume functions

Today I would repeat the content of the previous lecture, taking advantage of the material we have covered in September assignments.

DEFINITION: Let $\mathbf{C}$ be the set of compact subsets in a topological space M. A function $\lambda: \mathbf{C} \longrightarrow \mathbb{R}^{\geqslant 0}$ is

* **Monotone**, if $\lambda(A) \leqslant \lambda(B)$ for $A \subset B$
* **Additive**, if $\lambda(A \coprod B) = \lambda(A) + \lambda(B)$
* **Semiadditive**, if $\lambda(A \cup B) \leqslant \lambda(A) + \lambda(B)$

If these assumptions are satisfied, λ is called **volume function**.

DEFINITION: Let λ be a volume on M. For any $S \subset M$, define **inner measure** $\lambda_*(S) := \sup_C \lambda(C)$, where supremum is taken over all compact $C \subset S$, and **outer measure** $\lambda^*(S) := \inf_U \lambda_*(U)$, where infimum is taken over all open $U \supset S$.

DEFINITION: A volume is called **regular** if $\lambda^*(S) = \lambda(S)$ for any compact subset $S \subset M$.

Radon measures

DEFINITION: Radon measure. or **regular measure** on a locally compact topological space M is a Borel measure μ which satisfies the following assumptions.

1. μ is finite on all compact sets.
2. For any Borel set E, one has $\mu(E) = \inf \mu(U)$, where infimum is taken over all open U containing E.
3. For any open set E, one has $\mu(E) = \sup \mu(K)$, where infimum is taken over all compact K contained in E.

THEOREM: Outer measure is always a Radon measure.

Proof: Assignment 6. ■

Riesz representation theorem

DEFINITION: Uniform topology on functions is induced by the metric $d(f,g) = \sup|f-g|$.

Riesz representation theorem: Let M be a metrizable, locally compact topological space, and $C_c^0(M)^*$ the space of functionals continuous in uniform topology. **Then Radon measures can be characterized as continuous functionals $\mu \in C_c^0(M)^*$ which are non-negative on all non-negative functions.**

Proof: Clearly, all measures define such functionals. Conversely, let $\rho \in C_c^0(M)^*$ be a functional which is non-negative on non-negative functions. Given a compact set $K \subset M$, denote by χ_K its **characteristic function**, that is, a function which is equal 1 on K and 0 on $M \backslash K$. Consider the number $\lambda(K) := \rho(f)$, where the infimum is taken over all functions $f \in C_c^0(M)$ such that $f \geqslant \chi_K$. This function is clearly subadditive and monotonous. It is additive because for any two non-intersecting compact sets there exists a continuous function taking 0 on one and 1 on another **(prove it)**. The corresponding outer measure μ satisfies $\rho(f) = \int_M f\mu$ **(prove it).** ■

Weak-$*$ topology (reminder)

DEFINITION: Let M be a topological space, and $C^0_c(M)$ the space of continuous function with compact support. Any finite Borel measure μ defines a functional $C^0_c(M) \longrightarrow \mathbb{R}$ mapping f to $\int_M f\mu$. We say that a sequence $\{\mu_i\}$ of measures **converges in weak-$*$ topology** (or **in measure topology**) to μ if

$$\lim_i \int_M f\mu_i = \int_M f\mu$$

for all $f \in C^0_c(M)$. **The base of open sets of weak-$*$ topology** is given by $U_{f,]a,b[}$ where $]a,b[\subset \mathbb{R}$ is an interval, and $U_{f,]a,b[}$ is the set of all measures μ such that $a < \int_M f\mu < b$.

Tychonoff topology (reminder)

DEFINITION: Let $\{X_\alpha\}$ be a family of topological spaces, parametrized by $\alpha \in \mathcal{I}$. **Product topology**, or **Tychonoff topology** on the product $\prod_\alpha X_\alpha$ is topology where the open sets are generated by unions and finite intersections of $\pi_a^{-1}(U)$, where $\pi_a : \prod_\alpha X_\alpha$ is a projection to the X_a-component, and $U \subset X_a$ is an open set.

REMARK: Tychonoff topology is also called **topology of pointwise convergence**, because the points of $\prod_\alpha X_\alpha$ can be considered as maps from the set of indices $\mathcal{I}$ to the corresponding X_α, and a sequence of such maps converges if and only if it converges for each $\alpha \in \mathcal{I}$.

REMARK: Consider a finite measure as an element in the product of $C_c^0(M)$ copies of $\mathbb{R}$, that is, as a continuous map from $C_c^0(M)$ to $\mathbb{R}$. **Then the weak-$*$ topology is induced by the Tychonoff topology on this product.**

Space of measures and Tychonoff topology (reminder)

REMARK: (**Tychonoff theorem**)
A product of any number of compact spaces is compact.

THEOREM: Let M be a compact topological space, and $\mathcal{P}$ the space of probability measures on M equipped with the measure topology. **Then $\mathcal{P}$ is compact.**

Proof. Step 1: For any probability measure on M, and any $f \in C_c^0(M)$, one has $\min(f) \leqslant \int_M f\mu \leqslant \max(f)$. Therefore, μ can be considered as an element of the product $\prod_{f \in C_c^0(M)}[\min(f), \max(f)]$ of closed intervals indexed by $f \in C_c^0(M)$, and **Tychonoff topology on this product induces the weak-$*$ topology.**

Step 2: A closed subset of a compact set is again compact, hence **it suffices to show that all limit points of $\mathcal{P} \subset \prod_{f \in C_c^0(M)}[\min(f), \max(f)]$ are probability measures.** This is implied by Riesz representation theorem. The limit measure μ satisfies $\mu(M) = 1$ because the constant function $f = 1$ has compact support, hence $\lim \int_M \mu_i = \int_M \mu$ whenever $\lim_i \mu_i = \mu$. It is continuous because $\mu(f) \leqslant \varepsilon$ for any function taking values in $[0, \varepsilon]$. ■

Fréchet spaces

DEFINITION: A seminorm on a vector space V is a function $\nu: V \longrightarrow \mathbb{R}^{\geqslant 0}$ satisfying

1. $\nu(\lambda x) = |\lambda|\nu(x)$ for each $\lambda \in \mathbb{R}$ and all $x \in V$

2. $\nu(x+y) \leqslant \nu(x) + \nu(y)$.

DEFINITION: We say that **topology on a vector space V is defined by a family of seminorms $\{\nu_\alpha\}$** if the base of this topology is given by the finite intersections of the sets

$$B_{\nu_\alpha,\varepsilon}(x) := \{y \in V \quad | \quad \nu_\alpha(x-y) < \varepsilon\}$$

("open balls with respect to the seminorm"). It is **complete** if each sequence $x_i \in V$ which is Cauchy with respect to each of the seminorms converges.

DEFINITION: A **Fréchet space** is a Hausdorff second countable topological vector space V with the topology defined by a countable family of seminorms, complete with respect to this family of seminorms.

Seminorms and weak-$*$ topology

REMARK: Let M be a manifold and W be the subspace in functionals on $C^0_c(M)$ generated by all Borel measures ("the space of signed measures"). Recall that **the Hahn decomposition** is a decomposition of $\mu \in W$ as $\mu = \mu_+ - \mu_-$, where μ_+, μ_- are measures with non-intersecting support.

EXAMPLE: Then the weak-$*$ topology is defined by a countable family of seminorms. Indeed, we can choose a dense, countable family of functions $f_i \in C^0_c(M)$, and define the seminorms ν_{f_i} on measures by $\nu_{f_i}(\mu) := \int_M f_i\mu$ extending it to W by $\nu_{f_i}(\mu) = \int_M f_i\mu_+ + \int_M f_i\mu_-$, where $\mu = \mu_+ - \mu_-$ is the Hahn decomposition.

Existence of invariant measures

Further on, we shall prove the following theorem

Theorem 1: Let $K \subset V$ be a compact, convex subset of a topological vector space with topology defined by a family of seminorms, and $A: V \longrightarrow V$ a continuous linear map which preserves K. **Then there exists a point $z \in K$ such that $A(z) = z$.**

Its proof is in the next slide.

COROLLARY: Let M be a compact topological space and $f: M \longrightarrow M$ a continuous map. **Then there exists an f-invariant probability measure on M.**

Proof: Take the compact space $K \subset W$ of all probability measures, and let $A: K \longrightarrow K$ map μ to $f_* \mu$. Then A has a fixed point, as follows from Theorem 1. ∎

Linear maps on convex compact sets

Theorem 1: Let $K \subset V$ be a compact, convex subset of a topological vector space with topology defined by a family of seminorms, and $A: V \longrightarrow V$ a continuous linear map which preserves K. **Then there exists a point $z \in K$ such that $A(z) = z$.**

Proof: Consider the linear map $A_n(x) := \frac{1}{n}\sum_{i=0}^{n-1} A^n(x)$. Since it is an average of points in K, one has $A_n(x) \in K$. Let $z \in K$ be a limit point of the sequence $\{A_n(x)\}$ for some $x \in K$. Since

$$(1-A)A_n(x) = \frac{(1-A)\left(\sum_{i=0}^{n-1} A^n\right)}{n} = \frac{1-A^n}{n},$$

for each seminorm ν_i on V one has

$$\nu(A(A_n(x)) - A_n(x)) < \frac{C}{n},$$

where

$$C := \sup_{x,y\in K} \nu(x-y).$$

By continuity of ν, this gives $\nu(A(z)-z) < \frac{C}{n}$ for each $n > 0$, hence $A(z) = z$.
■

Linear maps on convex compact sets: properties of the limit

Lemma 1: Let $K \subset V$ be a compact, convex subset of a topological vector space with topology defined by a family of seminorms, and $A: V \longrightarrow V$ a continuous linear map which preserves K. Consider the map $A_n(x) := \frac{1}{n}\sum_{i=0}^{n-1} A^i(x)$, and let $\operatorname{Map}(K,K)$ be the space of maps from K to itself with the Tychonoff topology. Then $\{A_n\}$ has a subsequence converging to a linear map B from K to itself. Consider B as a linear map from the space $V' \subset V$ generated by K to itself. **Then for two such limits B_1 and B_2, the difference $E := B_1 - B_2$ satisfies $\operatorname{im} E \subset V_0, \ker E \subset V_0$, where $V_0 = \ker(1 - A) \cap V'$.**

Proof. Step 1: Consider the space $\operatorname{Map}(K,K)$ of maps from K to itself with the product topology. By Tychonoff theorem, it is compact. The set of linear maps is closed in $\operatorname{Map}(K,K)$ **(prove it)**. Then **the sequence $\{A_n \in \operatorname{Map}(K,K)\}$ has a limit point** $B: K \longrightarrow K$ which is a linear map on K. Then **B defines a linear (possibly discontinuous) endomorphism of V'.**

Step 2: Since $(1-A)A_n(x) = \frac{1-A^n}{n}$, one has $(1-A)B = B(1-A) = 0$. This implies that $\operatorname{im} B \subset V_0$. Since $B\big|_{V_0} = A$, we also have $E\big|_{V_0} = V_0$. ■

Measures with linear bound

Lemma 2 Let $C > 0$ be a constant, ν a measure on S, and $K_{C,\nu}$ be the space of measures μ on S which satisfy $\mu(U) \leqslant C\nu(U)$. for all measurable sets U. **Then K_ν is closed in weak-$*$ topology.**

Proof: $K_{C,\nu} = \bigcap_{f\in C^0_c(M)} K_f$, where $K_f = \{\text{measures } \mu \mid \int_S |f|\mu \leqslant C\int_S |f|\nu.\}$ ■

Birkhoff-Khinchin Ergodic Theorem

THEOREM: (Birkhoff-Hinchin Ergodic Theorem) Let $f: M \longrightarrow M$ be a continuous map on a compact topological space, and μ a probability measure. Assume that $\mu = \Phi\nu$, where $f_*\nu = \nu$, and $|\Phi| < C$ a bounded measurable function. **Then the sequence $\mu_n := \frac{1}{n}\sum_{i=0}^{n-1}(f_*)^i\mu$ converges to a probability measure.**

Proof. Step 1: The sequence $\mu_n := \frac{1}{n}\sum_{i=0}^{n-1}(f_*)^i\mu$ has a limit point μ' which is absolutely continuous with respect to ν by Lemma 2. Moreover, the function $\Psi := \frac{\mu'}{\nu}$ is bounded by the same constant C. Since $|\mu_n - f_*\mu_n| < \frac{|\mu_n| - |f_*^n\mu_n|}{n}$, **the limit function Ψ is f-invariant.**

Step 2: Consider the map $E: K \longrightarrow V_0$ of Lemma 1, where K is the space of probability measures. Using the natural pairing $f, g \longrightarrow \int_M fg\mu$, we embed the space $C^0_c(M)$ to $C^0_c(M)^*$. Then E can be interpreted as an f_*-invariant V_0-valued functional $Z: C^0_c(M) \longrightarrow V_0$, vanishing on all functions which have measure 0 with respect to μ.

Composing Z with a linear functional κ, and applying Radon-Nikodym theorem, we obtain an integrable f_*-invariant function $\Theta \in L^1(M)$ such that $\kappa(Z(\Phi\mu)) = \int_M \Theta\Phi\mu$. Then $Z(\Theta) \neq 0$, because $Z(\Theta) = \int_M \Theta^2 > 0$. This is impossible, because $E|_{V\,0} = 0$. ■

Hopf Argument

DEFINITION: Let M be a metric space with a Borel measure and $F : M \longrightarrow M$ a continuous map preserving measure. The **"stable foliation"** is an equivalence relation on M, with $x\tilde{y}$ when $\lim_i d(F^n(x), F^n(y)) = 0$. The **"leaves"** of stable foliation are equivalence classes.

THEOREM: (Hopf Argument) Any measurable, F-invariant function **is constant on the leaves of stable foliation** outside of a measure 0 set.

Proof: Let $A(f) := \lim_n \frac{1}{n} \sum_{i=0}^{n-1} (F^i)^* f$ be the map provided by Birkhoff-Khinchin theorem. It suffices to prove that $A(f)$ is constant only for the functions in $\operatorname{im} A$. Since Lipschitz functions are dense in L^1-topology, it suffices to show this only when f is C-Lipschitz for some $C > 0$.

For any sequence $\alpha_i \in \mathbb{R}$ converging to 0, the sequence $\frac{1}{n} \sum_{i=0}^{n-1} \alpha_i$ also converges to 0. Therefore, whenever $x\tilde{y}$, one has

$$A(f)(x) - A(f)(y) = \lim_n \sum_{i=0}^{n-1} f(F^i(x)) - f(F^i(y)) = 0$$

because $\alpha_i = |f(F^i(x)) - f(F^i(y))| \leqslant C d(F^i(x), F^i(y))$ converges to 0. ■