Teoria Ergódica Diferenciável

lecture 9: Conformal automorphisms of a disc

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Riemannian manifolds (reminder)

DEFINITION: Let $h \in \operatorname{Sym}^2 T^*M$ be a symmetric 2-form on a manifold which satisfies $h(x,x) > 0$ for any non-zero tangent vector x. Then h is called **Riemannian metric**, of **Riemannian structure**, and (M,h) **Riemannian manifold**.

DEFINITION: For any $x, y \in M$, and any piecewise smooth path $\gamma : [a,b] \longrightarrow M$ connecting x and y, consider **the length** of γ defined as $L(\gamma) = \int_\gamma |\frac{d\gamma}{dt}| dt$, where $|\frac{d\gamma}{dt}| = h(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})^{1/2}$. Define **the geodesic distance** as $d(x,y) = \inf_\gamma L(\gamma)$, where infimum is taken for all paths connecting x and y.

EXERCISE: Prove that the **geodesic distance satisfies triangle inequality and defines a metric on M.**

EXERCISE: Prove that **this metric induces the standard topology on M.**

EXAMPLE: Let $M = \mathbb{R}^n$, $h = \sum_i dx_i^2$. **Prove that the geodesic distance coincides with $d(x,y) = |x-y|$.**

EXERCISE: Using partition of unity, **prove that any manifold admits a Riemannian structure.**

Conformal structures (reminder)

DEFINITION: Let h, h' be Riemannian structures on M. These Riemannian structures are called **conformally equivalent** if $h' = fh$, where f is a positive smooth function.

DEFINITION: **Conformal structure** on M is a class of conformal equivalence of Riemannian metrics.

DEFINITION: **A Riemann surface** is a 2-dimensional oriented manifold equipped with a conformal structure.

DEFINITION: Let $I : TM \longrightarrow TM$ be an endomorphism of a tangent bundle satisfying $I^2 = -\operatorname{Id}$. Then I is called **almost complex structure operator**, and the pair (M, I) **an almost complex manifold**.

CLAIM: Let M be a 2-dimensional oriented conformal manifold. **Then M admits a unique orthogonal almost complex structure** in such a way that the pair $x, I(x)$ is positively oriented. Conversely, **an almost complex structure uniquely determines the conformal structure nd orientation.**

Homogeneous spaces (reminder)

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \longrightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\operatorname{St}_x(G) = \{g \in G \mid g(x) = x\}$ is called **stabilizer of a point** x, or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G, **one has $M = G/H$,** where $H = \operatorname{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \longrightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H. ■

REMARK: Let $g(x) = y$. Then $\operatorname{St}_x(G)^g = \operatorname{St}_y(G)$: **all the isotropy groups are conjugate.**

Isotropy representation (reminder)

DEFINITION: Let $M = G/H$ be a homogeneous space, $x \in M$ and $\mathrm{St}_x(G)$ the corresponding stabilizer group. The **isotropy representation** is the natural action of $\mathrm{St}_x(G)$ on T_xM.

DEFINITION: A Riemannian form Φ on a homogeneous manifold $M = G/H$ is called **invariant** if it is mapped to itself by all diffeomorphisms which come from $g \in G$.

REMARK: Let Φ_x be an isotropy invariant scalar product on T_xM. For any $y \in M$ obtained as $y = g(x)$, consider the form Φ_y on T_yM obtained as $\Phi_y := g(\Phi)$. The choice of g is not unique, however, for another $g' \in G$ which satisfies $g'(x) = y$, we have $g = g'h$ where $h \in \mathrm{St}_x(G)$. Since Φ_x is h-invariant, **the metric Φ_y is independent from the choice of g.**

We proved

THEOREM: **Homogeneous Riemannian forms on $M = G/H$ are in bijective correspondence with isotropy invariant spalar products on T_xM,** for any $x \in M$. ■

Space forms (reminder)

DEFINITION: Simply connected space form is a homogeneous manifold of one of the following types:

positive curvature: S^n (an n-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: $\mathbb{R}^n$ (an n-dimensional Euclidean space), equipped with an action of isometries

negative curvature: $SO(1,n)/O(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

Riemannian metric on space forms

LEMMA: Let $G = SO(n)$ act on $\mathbb{R}^n$ in a natural way. **Then there exists a unique G-invariant symmetric 2-form:** the standard Euclidean metric.

Proof: Let g, g' be two G-invariant symmetric 2-forms. Since S^{n-1} is an orbit of G, we have $g(x,x) = g(y,y)$ for any $x, y \in S^{n-1}$. Multiplying g' by a constant, we may assume that $g(x,x) = g'(x,x)$ for any $x \in S^{n-1}$. **Then $g(\lambda x, \lambda x) = g'(\lambda x, \lambda x)$ for any $x \in S^{n-1}$, $\lambda \in \mathbb{R}$;** however, all vectors can be written as λx. ■

COROLLARY: Let $M = G/H$ be a simply connected space form. **Then M admits a unique, up to a constant multiplier, G-invariant Riemannian form.**

Proof: The isotropy group is $SO(n-1)$ in all three cases, and the previous lemma can be applied. ■

REMARK: From now on, all space forms are assumed to be homogeneous Riemannian manifolds.

Poincaré-Koebe uniformization theorem

DEFINITION: A **Riemannian manifold of constant curvature** is a Riemannian manifold which is locally isometric to a space form.

THEOREM: **(Poincaré-Koebe uniformization theorem)** Let M be a Riemann surface. **Then M admits a unique complete metric of constant curvature in the same conformal class.**

COROLLARY: **Any Riemann surface is a quotient of a space form X by a discrete group of isometries $\Gamma \subset \operatorname{Iso}(X)$.**

COROLLARY: **Any simply connected Riemann surface is conformally equivalent to a space form.**

Lie groups and their properties

DEFINITION: Lie algebra of a Lie group G is the Lie algebra Lie(G) of left-invariant vector fields. **Adjoint representation** of G is the standard action of G on Lie(G). For a Lie group $G = GL(n)$, $SL(n)$, etc., $PGL(n)$, $PSL(n)$, etc. denote the image of G in GL(Lie(G)) with respect to the adjoint action.

REMARK: This is the same as a quotient G/Z by the centre of G.

EXERCISE: Prove that **the center of $PSL(n, \mathbb{R})$, $PSO(n, \mathbb{R})$, etc. is trivial.**

EXERCISE: Prove that a **discrete normal subgroup of $SL(n, \mathbb{R})$ is central** (commutes with everything).

EXERCISE: Let $\Psi : G \longrightarrow G_1$ be a homomorphism of connected Lie groups of the same dimension with $d\Psi$ surjective. Prove that **Ψ is a covering** (quotient by a discrete subgroup).

Hint: Use the inverse function theorem.

Some low-dimensional Lie group isomorphisms

DEFINITION: Let $SO^+(1,2)$ be the connected component of the group of orthogonal matrices on a 3-dimensional space equipped with a scalar product of signature (1,2), and $U(1,1)$ the group of complex linear maps $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ preserving a pseudio-Hermitian form of signature (1,1).

THEOREM: The groups $PU(1,1)$, $PSL(2,\mathbb{R})$ and $SO^+(1,2)$ are isomorphic.

Proof: Isomorphism $PU(1,1) = SO^+(1,2)$ will be established later. To see $PSL(2,\mathbb{R}) \cong SO^+(1,2)$, consider **the Killing form** κ on the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, $a, b \longrightarrow \operatorname{Tr}(ab)$. **Check that it has signature $(1,2)$. Then the image of $SL(2,\mathbb{R})$ in automorphisms of its Lie algebra is $SO(\mathfrak{sl}(2,\mathbb{R}),\kappa) = SO^+(1,2)$.** Both groups are 3-dimensional, and differential of the map

$$\Psi: \ PSL(2,\mathbb{R}) \longrightarrow SO^+(1,2)$$

is an isomorphism. Then **Ψ is surjective and has discrete kernel.** However, the kernel subgroup has to be central, and **$PSL(2,\mathbb{R})$ has no center** by construction. ■

Holomorphic functions

DEFINITION: Let $I: TM \longrightarrow TM$ be an endomorphism of a tangent bundle satisfying $I^2 = -\operatorname{Id}$. Then I is called **almost complex structure operator**, and the pair (M, I) **an almost complex manifold**.

EXAMPLE: $M = \mathbb{C}^n$, with complex coordinates $z_i = x_i + \sqrt{-1}\, y_i$, and $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$.

DEFINITION: A function $f: M \longrightarrow \mathbb{C}$ on an almost complex manifold is called **holomorphic** if df is $\mathbb{C}$-linear.

Holomorphic functions on $\mathbb{C}^n$

THEOREM: Let $f: M \longrightarrow \mathbb{C}$ be a differentiable function on an open subset $M \subset \mathbb{C}$, with the natural almost complex structure. **Then the following are equivalent.**

(1) f **is holomorphic**.

(2) f **is conformal** in all points where df is non-zero, and preserves the orientation.

(3) f is expressed as a sum of Taylor series around any point $z \in M$:

$$f(z+t) = \sum_{i=0}^{n} \frac{f^{(i)}(z)t^i}{i!}$$

(here we assume that the complex number t satisfies $|t| < \varepsilon$, where ε depends on f and z).

Proof: Taylor series decomposition on a line is implied by the Cauchy formula:

$$\int_{\partial\Delta} \frac{f(z)dz}{z-a} = 2\pi\sqrt{-1}\, f(a),$$

where $\Delta \subset \mathbb{C}$ is a disk, $a \in \Delta$ any point, and z coordinate on $\mathbb{C}$. Indeed, in this case, $2\pi\sqrt{-1}\, f(a) = \sum_{i \geqslant 0} a^i \int_{\partial\Delta} f(z)(z^{-1})^{i+1}$, because $\frac{1}{z-a} = z^{-1}\sum_{i\geqslant 0}(az^{-1})^i$.

■

Cauchy formula

Let's prove Cauchy formula, using Stokes' theorem. Since the space $\mathbb{C}$-linear 1-forms on $\mathbb{C}$ is 1-dimensional, $df \wedge dz = 0$ for any holomorphic function on $\mathbb{C}$. This gives

CLAIM: A function on a disk $\Delta \subset \mathbb{C}$ **is holomorphic if and only if the form $\eta := f dz$ is closed** (that is, satisfies $d\eta = 0$). ■

Now, let S_ε be a radius ε circle around a point $a \in \Delta$, Δ_ε its interior, and $\Delta_0 := \Delta \backslash \Delta_\varepsilon$. Stokes' theorem gives

$$0 = \int_{\Delta_0} d\left(\frac{f(z)dz}{z-a}\right) = -\int_{S_\varepsilon} \frac{f(z)dz}{z-a} + \int_{\partial\Delta} \frac{f(z)dz}{z-a},$$

hence Cauchy formula would follow if we show that $\lim_{\varepsilon \to 0} \int_{S_\varepsilon} \frac{f(z)dz}{z-a} = 2\pi\sqrt{-1}f(a)$.

Assuming for simplicity $a = 0$ and parametrizing the circle S_ε by $\varepsilon e^{\sqrt{-1}\,t}$, we obtain

$$\int_{S_\varepsilon} \frac{f(z)dz}{z} = \int_0^{2\pi} \frac{f(\varepsilon e^{\sqrt{-1}\,t})}{\varepsilon e^{\sqrt{-1}\,t}} d(\varepsilon e^{\sqrt{-1}\,t}) =$$
$$= \int_0^{2\pi} \frac{f(\varepsilon e^{\sqrt{-1}\,t})}{\varepsilon e^{\sqrt{-1}\,t}} \sqrt{-1}\,\varepsilon e^{\sqrt{-1}\,t} dt = \int_0^{2\pi} f(\varepsilon e^{\sqrt{-1}\,t})\sqrt{-1}\,dt$$

as ε tends to 0, $f(\varepsilon e^{\sqrt{-1}\,t})$ tends to $f(0)$, and this integral goes to $2\pi\sqrt{-1}f(0)$.

Schwartz lemma

CLAIM: (maximum principle) Let f be a holomorphic function defined on an open set U. **Then f cannot have strict maxima in U. If f has non-strict maxima, it is constant.**

EXERCISE: Prove the maximum principle.

LEMMA: (Schwartz lemma) Let $f: \Delta \longrightarrow \Delta$ be a map from disk to itself fixing 0. **Then $|f'(0)| \leqslant 1$, and equality can be realized only if $f(z) = \alpha z$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$.**

Proof: Consider the function $\varphi := \frac{f(z)}{z}$. Since $f(0) = 0$, it is holomorphic, and since $f(\Delta) \subset \Delta$, on the boundary $\partial\Delta$ we have $|\varphi||_{\partial\Delta} \leqslant 1$. Now, **the maximum principle implies that $|f'(0)| = |\varphi(0)| \leqslant 1$,** and equality is realized only if $\varphi = \text{const}$. ■