Teoria Ergódica Diferenciável

lecture 11: Möbius group

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Laurent power series

DEFINITION: Laurent power series is a function expressed as $f(z) = \sum_{i\in\mathbb{Z}} z^i a_i$

REMARK: A holomorphic function $\varphi : \mathbb{C}^* \longrightarrow \mathbb{C}$ uniquely determines its Laurent power series. Indeed, residue of $z^k\varphi$ in 0 is $\sqrt{-1}\, 2\pi a_{-k-1}$.

Laurent power series: function in an annulus

THEOREM: (Laurent theorem)
Let f be a holomorphic function on an annulus (that is, a ring)

$$R = \{z \mid \alpha < |z| < \beta\}.$$

Then f can be expressed as a Laurent power series $f(z) = \sum_{i\in\mathbb{Z}} z^i a_i$ converging in R.

Proof: Same as Cauchy formula: for an annulus with components of the boundary denoted as ∂R_+ and ∂R_-, one has

$$\int_{\partial R_+} \frac{f(z)dz}{z-a} - \int_{\partial R_-} \frac{f(z)dz}{z-a} = 2\pi\sqrt{-1}\, f(a),$$

This gives

$$2\pi\sqrt{-1}\, f(a) = \sum_{i\geqslant 0} a^i \int_{\partial R_+} f(z)(z^{-1})^{i+1} - \sum_{i\geqslant 0} a^{-i-1} \int_{\partial R_-} f(z)z^i$$

because $\frac{1}{z-a} = z^{-1}\sum_{i\geqslant 0}(az^{-1})^i$ for $|z| > |a|$ and $\frac{1}{z-a} = a^{-1}\sum_{i\geqslant 0}(a^{-1}z)^i$ for $|z| < |a|$. ■

REMARK: This theorem remains valid if $\alpha = 0$ and $\beta = \infty$.

Affine coordinates on $\mathbb{C}P^1$

DEFINITION: We identify $\mathbb{C}P^1$ with the set of pairs $x:y$ defined up to equivalence $x:y \sim \lambda x:\lambda y$, for each $\lambda \in \mathbb{C}^*$. This representation is called **homogeneous coordinates**. **Affine coordinates** are $1:z$ for $x \neq 0$, $z = y/x$ and $z:1$ for $y \neq 0$, $z = x/y$. The corresponding gluing functions are given by the map $z \longrightarrow z^{-1}$.

DEFINITION: **Meromorphic function** is a quotient f/g, where f, g are holomorphic and $g \neq 0$.

REMARK: A holomorphic map $\mathbb{C} \longrightarrow \mathbb{C}P^1$ is the same as a pair of maps $f:g$ up to equivalence $f:g \sim fh:gh$. **In other words, holomorphic maps $\mathbb{C} \longrightarrow \mathbb{C}P^1$ are identified with meromorphic functions on $\mathbb{C}$.**

REMARK: In homogeneous coordinates, an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{C})$ acts as $x:y \longrightarrow ax+by : cx+dy$. Therefore, in affine coordinates it acts as $z \longrightarrow \frac{az+b}{cz+d}$.

Möbius transforms

DEFINITION: **Möbius transform** is a conformal (that is, holomorphic) diffeomorphism of $\mathbb{C}P^1$. **Möbius group** is the group of Möbius transforms.

REMARK: The group $PGL(2, \mathbb{C})$ acts on $\mathbb{C}P^1$ holomorphially.

The following theorem will be proven in the next slide.

THEOREM: The natural map from $PGL(2, \mathbb{C})$ to the group of Möbius transforms is an isomorphism.

REMARK: Let $\varphi : \mathbb{C}^* \longrightarrow \mathbb{C}$ be a holomorphic function, and $\varphi = \sum_{i \in \mathbb{Z}} z^i a_i$ its Laurent power series. **Then $\psi(z) := \varphi(z^{-1})$ has Laurent polynomial $\psi = \sum_{i \in \mathbb{Z}} z^{-i} a_i$.**

This implies

Claim 1: Let $\varphi : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a holomorphic automorphism, $\varphi_0 : \mathbb{C} \longrightarrow \mathbb{C}P^1$ its restriction to the chart $z : 1$, and $\varphi_\infty : \mathbb{C} \longrightarrow \mathbb{C}P^1$ its restriction $1 : z$. We consider $\varphi_0, \varphi_\infty$ as meromorphic functions on $\mathbb{C}$. **Then $\varphi_\infty = \varphi_0(z^{-1})^{-1}$.**

Möbius transforms and $PGL(2,\mathbb{C})$

THEOREM: The natural map from $PGL(2,\mathbb{C})$ to the group $\operatorname{Aut}(\mathbb{C}P^1)$ of Möbius transforms is an isomorphism.

Proof. Step 1: Let $\varphi \in \operatorname{Aut}(\mathbb{C}P^1)$. Since $PSL(2,\mathbb{C})$ acts transitively on pairs of points $x \neq y$ in $\mathbb{C}P^1$, by composing φ with an appropriate element in $PGL(2,\mathbb{C})$ we can assume that $\varphi(0)=0$ and $\varphi(\infty)=\infty$. This means that we may consider the restrictions φ_0 and φ_∞ of φ to the affine charts as a holomorphic functions on these charts, $\varphi_0, \varphi_\infty : \mathbb{C} \longrightarrow \mathbb{C}$.

Step 2: Let $\varphi_0 = \sum_{i>0} a_i z^i$, $a_1 \neq 0$. Claim 1 gives

$$\varphi_\infty(z) = \varphi_0(z^{-1})^{-1} = a_1 z(1 + \sum_{i\geqslant 2} \frac{a_i}{a_1} z^{-i})^{-1}.$$

Unless $a_i = 0$ for all $i \geqslant 2$, this Laurent series has singularities in 0 and cannot be holomorphic. **Therefore φ_0 is a linear function**, and it belongs to $PGL(2,\mathbb{C})$. ∎

Lemma 1: Let φ be a Möbius transform fixing $\infty \in \mathbb{C}P^1$. **Then $\varphi(z) = az+b$ for some $a,b \in \mathbb{C}$ and all $z = z:1 \in \mathbb{C}P^1$.**
Proof: Let $A \in PGL(2,\mathbb{C})$ be a map acting on $\mathbb{C} = \mathbb{C}P^1 \backslash \infty$ as parallel transport mapping $\varphi(0)$ to 0. Then $\varphi \circ A$ is a Moebius transform which fixes ∞ and 0. As shown in Step 2 above, it is a linear function. ∎

Circles in $\mathbb{C}P^1$

DEFINITION: A circle in S^2 is an orbit of a 1-parametric subgroup $S^1 \subset GL(2,\mathbb{C})$.

REMARK: Any subgroup $S^1 \subset PGL(2,\mathbb{C})$ acts by isometry for an appropriate Hermitian metric. Indeed, we can pick any Hermitian metric on $\mathbb{C}^2$ and average it with the S^1-action.

REMARK: Consider a pseudo-Hermitian form h on $V = \mathbb{C}^2$ of signature (1,1). Let h_+ be a positive definite Hermitian form on V. There exists a basis $x, y \in V$ such that $h_+ = \sqrt{-1}\, x \otimes \overline{x} + \sqrt{-1}\, y \otimes \overline{y}$ (that is, x, y is orthonormal with respect to h_+) and $h = -\sqrt{-1}\,\alpha x \otimes \overline{x} + \sqrt{-1}\,\beta y \otimes \overline{y}$, with $\alpha > 0$, $\beta < 0$ real numbers. Then $\{z \mid h(z,z) = 0\}$ is invariant under the rotation $x, y \longrightarrow x, e^{\sqrt{-1}\,\theta} y$, hence **it is a circle.**

Möbius transform preserves circles

REMARK: We have just shown that **the zero set of a pseudo-Hermitian form is a circle in $\mathbb{C}P^1$.**

LEMMA: All circles $S \subset \mathbb{C}P^1$ can be obtained this way.

Proof: Using exponent map and the the Jordan normal form, we obtain that $S^1 \subset GL(2,\mathbb{C})$ can be given by a matrix

$$\rho(t) = \begin{pmatrix} e^{\sqrt{-1}\,\pi n t} & 0 \\ 0 & e^{\sqrt{-1}\,\pi m t} \end{pmatrix},$$

for some $n, m \in \mathbb{Z}$. Let z_1, z_2 be the corresponding coordinates on $\mathbb{C}^2$. Choose $h = a|z_1|^2 - b|z_2|^2$ in such a way that $h(z) := h(z, \overline{z}) = 0$ for some $z \in S$. Then $h|_S = 0$. The set of points $v \in \mathbb{C}P^1$ such that $h(v) = 0$ is a circle, hence $S = \{v \in \mathbb{C}P^1 \ \ | \ \ h(v) = 0\}$. ■

PROPOSITION: The action of $PGL(2,\mathbb{C})$ on $\mathbb{C}P^1$ maps circles to circles.

Proof: Any matrix $A \in GL(2,\mathbb{C})$ maps pseudo-Hermitian forms to pseudo-Hermitian forms, hence it maps their zero sets to their zero sets. However, the zero sets of pseudo-Hermitian forms are circles, as shown above. ■

Some low-dimensional Lie group isomorphisms

DEFINITION: For a Lie group such G as $GL(n)$, $SL(n)$, $U(p,q)$, ... denote by $PGL(n)$, $PSL(n)$, $PU(p,q)$, the quotient G/Z, where Z is the center of G.

DEFINITION: Let $SO^+(1,2)$ be the connected component of the group of orthogonal matrices on a 3-dimensional space equipped with a scalar product of signature (1,2), and $U(1,1)$ the group of complex linear maps $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ preserving a pseudio-Hermitian form of signature (1,1).

THEOREM: The groups $PU(1,1)$, $PSL(2,\mathbb{R})$ and $SO^+(1,2)$ are isomorphic.

Proof: Isomorphism $PU(1,1) = SO^+(1,2)$ will be established later today. To see $PSL(2,\mathbb{R}) \cong SO^+(1,2)$, consider **the Killing form** κ on the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, $a, b \longrightarrow \operatorname{Tr}(ab)$. **Check that it has signature $(1,2)$. Then the image of $SL(2,\mathbb{R})$ in automorphisms of its Lie algebra is $SO(\mathfrak{sl}(2,\mathbb{R}),\kappa) = SO^+(1,2)$.** Both groups are 3-dimensional, and differential of the map

$$\Psi: \ PSL(2,\mathbb{R}) \longrightarrow SO^+(1,2)$$

is an isomorphism. Then **Ψ is surjective and has discrete kernel.** However, the kernel subgroup has to be central, and **$PSL(2,\mathbb{R})$ has no center** by construction. ■

Transitive action is determined by a stabilizer of a point (reminder)

Lemma 2: Let $M = G/H$ be a homogeneous space, and $\Psi: \ G_1 \longrightarrow G$ a homomorphism such that G_1 acts on M transitively and $\operatorname{St}_x(G_1) = \operatorname{St}_x(G)$. **Then $G_1 = G$.**

Proof: Since any element in $\ker\Psi$ belongs to $\operatorname{St}_x(G_1) = \operatorname{St}_x(G) \subset G$, the homomorphism Ψ is injective. It remais only to show that Ψ is surjective.

Let $g \in G$. Since G_1 acts on M transitively, $gg_1(x) = x$ for some $g_1 \in G_1$. Then $gg_1 \in \operatorname{St}_x(G_1) = \operatorname{St}_x(G) \subset \operatorname{im} G_1$. This gives $g \in G_1$. ■

Group of conformal automorphisms of the disk is $PU(1,1)$ (reminder)

REMARK: The group $PU(1,1) \subset PGL(2,\mathbb{C})$ of unitary matrices preserving a pseudo-Hermitian form h of signature (1,1) acts on a disk $\{l \in \mathbb{C}P^1 \mid h(l,l) > 0\}$ by holomorphic automorphisms. Indeed, $PGL(2,\mathbb{C})$ acts conformally on $\mathbb{C}P^1$.

COROLLARY: Let $\Delta \subset \mathbb{C}$ be the unit disk, $\operatorname{Aut}(\Delta)$ the group of its conformal automorphisms, and $\Psi: PU(1,1) \longrightarrow \operatorname{Aut}(\Delta)$ the map constructed above. **Then Ψ is a group isomorphism.**

Proof: We use Lemma 2. Both groups act on Δ transitively, hence **it suffices only to check that $\operatorname{St}_x(PU(1,1)) = S^1$ and $\operatorname{St}_x(\operatorname{Aut}(\Delta)) = S^1$.** The first isomorphism is clear, because the space of unitary automorphisms fixing a vector v is $U(v^\perp)$. The second isomorphism follows from Schwartz lemma **(prove it!)**. ■

Upper half-plane

REMARK: The map $z \longrightarrow -\sqrt{-1}\,(z-1)^{-1}$ induces a diffeomorphism from the unit disc in $\mathbb{C}$ to the upper half-plane $\mathbb{H}$.

PROPOSITION: The group $\operatorname{Aut}(\Delta)$ acts on the upper half-plane $\mathbb{H}$ as $z \xrightarrow{A} \frac{az+b}{cz+d}$, where $a,b,c,d \in \mathbb{R}$, and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$.

REMARK: The group of such A is naturally identified with $PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$.

Proof: The group $PSL(2,\mathbb{R})$ preserves the line $\operatorname{im} z = 0$, hence acts on $\mathbb{H}$ by conformal automorphisms. The stabilizer of a point is S^1 **(prove it).** Now, Lemma 2 implies that $PSL(2,\mathbb{R}) = PU(1,1)$. ■

REMARK: We have shown that $\mathbb{H} = SO(1,2)/S^1$, hence **$\mathbb{H}$ is conformally equivalent to the hyperbolic space.**

Upper half-plane as a Riemannian manifold

DEFINITION: Poincaré half-plane is the upper half-plane equipped with an $PSL(2,\mathbb{R})$-invariant metric. By constructtion, **t is isometric to the Poincare disk and to the hyperbolic space form.**

THEOREM: Let (x, y) be the usual coordinates on the upper half-plane $\mathbb{H}$. **Then the Riemannian structure s on $\mathbb{H}$ is written as $s = \text{const}\frac{dx^2+dy^2}{y^2}$.**

Proof: Since the complex structure on $\mathbb{H}$ is the standard one and all Hermitian structures are proportional, we obtain that $s = \mu(dx^2+dy^2)$, where $\mu \in C^\infty(\mathbb{H})$. **It remains to find μ, using the fact that s is $PSL(2,\mathbb{R})$-invariant.**

For each $a \in \mathbb{R}$, the parallel transport $x \longrightarrow x + a$ fixes s, hence μ is a function of y. For any $\lambda \in \mathbb{R}^{>0}$, the map $H_\lambda(x) = \lambda x$, being holomorphic, also fixes s; since $\mathbb{H}_\lambda(dx^2 + dy^2) = \lambda^2 dx^2 + dy^2$, we have $\mu(\lambda x) = \lambda^{-2}\mu(x)$. ■

Geodesics on Riemannian manifold

DEFINITION: Minimising geodesic in a Riemannian manifold is a piecewise smooth path connecting x to y such that its length is equal to the geodesic distance. **Geodesic** is a piecewise smooth path γ such that for any $x \in \gamma$ there exists a neighbourhood of x in γ which is a minimising geodesic.

EXERCISE: Prove that a big circle in a sphere is a geodesic. Prove that an interval of a big circle of length $\leqslant \pi$ is a minimising geodesic.

Geodesics in Poincaré half-plane

THEOREM: Geodesics on a Poincaré half-plane are vertical straight lines and their images under the action of $SL(2,\mathbb{R})$.

Proof. Step 1: Let $a, b \in \mathbb{H}$ be two points satisfying $\operatorname{Re} a = \operatorname{Re} b$, and l the line connecting these two points. Denote by Π the orthogonal projection from $\mathbb{H}$ to the vertical line connecting a to b. For any tangent vector $v \in T_z\mathbb{H}$, one has $|D\pi(v)| \leqslant |v|$, and the equality means that v is vertical (prove it). Therefore, **a projection of a path γ connecting a to b to l has length $\leqslant L(\gamma)$, and the equality is realized only if γ is a straight vertical interval.**

Step 2: For any points a, b in the Poincaré half-plane, **there exists an isometry mapping (a, b) to a pair of points (a_1, b_1) such that $\operatorname{Re}(a_1) = \operatorname{Re}(b_1)$. (Prove it!)**

Step 3: Using Step 2, we prove that **any geodesic γ on a Poincaré half-plane is obtained as an isometric image of a straight vertical line:** $\gamma = v(\gamma_0)$, $v \in \operatorname{Iso}(\mathbb{H}) = PSL(2,\mathbb{R})$ ■

Geodesics in Poincaré half-plane are circles

CLAIM: Let S be a circle or a straight line on a complex plane $\mathbb{C} = \mathbb{R}^2$, and S_1 closure of its image in $\mathbb{C}P^1$ inder the natural map $z \longrightarrow 1 : z$. **Then S_1 is a circle, and any circle in $\mathbb{C}P^1$ is obtained this way.**

Proof: The circle $S_r(p)$ of radius r centered in $p \in \mathbb{C}$ is given by equation $|p - z| = r$, in homogeneous coordinates it is $|px - z|^2 = r|x|^2$. This is the zero set of the pseudo-Hermitian form $h(x, z) = |px - z|^2 - |x|^2$, hence it is a circle. ■

COROLLARY: Geodesics on the Poincaré half-plane are vertical straight lines and half-circles orthogonal to the line $\operatorname{im} z = 0$ in the intersection points.

Proof: We have shown that geodesics in the Poincaré half-plane are Möbius transforms of straight lines orthogonal to $\operatorname{im} z = 0$. However, any Möbius transform preserves angles and maps circles or straight lines to circles or straight lines. ■