Teoria Ergódica Diferenciável

lecture 15: Weak mixing: spectral characterization

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Koopman operators (reminder)

DEFINITION: Let (M,μ) be a space with finite measure, and $T: M \longrightarrow M$ a measurable map preserving measure. The triple (M,μ,T) is called **dynamical system**. The map T defines a isometric embedding T^* : Ł$^2(M,\mu) \longrightarrow L^2(M,\mu)$ on the space of square-integrable functions, called **the Koopman operator**.

DEFINITION: Two dynamical systems (M,μ,T) and (M_1,μ_1,T_1) are **spectral equivalent** if there exists an invertible map $\varphi: L^2(M,\mu) \longrightarrow L^2(M_1,\mu_1)$ such that the following diagram is commutative

$$\begin{array}{ccc} L^2(M,\mu) & \xrightarrow{\varphi} & L^2(M_1,\mu_1) \\ {\scriptstyle T^*}\downarrow & & \downarrow{\scriptstyle T_1^*} \\ L^2(M,\mu) & \xrightarrow{\varphi} & L^2(M_1,\mu_1) \end{array}$$

(this is the same as to say that the equivalence φ exchanges the Koopman operators T^* and T_1^*).

DEFINITION: A property A of dynamical system is called **spectral invariant** if for each two spectral invariant systems (M,μ,T) and (M_1,μ_1,T_1), the property A holds for (M,μ,T) $\Leftrightarrow$ it holds for (M_1,μ_1,T_1).

REMARK: We shall see today that **ergodicity is a spectral invariant.**

Ergodic measures and Cesàro sums (reminder)

Theorem (von Neumann ergodic): Let $U : H \longrightarrow H$ be an invertible orthogonal map, and $U_n := \frac{1}{n}\sum_{i=0}^{n-1} U^i(x)$. **Then $\lim_n U_n(x) = P(x)$, for all $x \in H$ where P is an orthogonal projection to $H^U = \ker(1-U)$.**

REMARK: a. e. means "almost everywhere", that is, outside of a measure 0 set.

THEOREM: Let (M,μ) be a space with (finite) measure, and $T : M \longrightarrow M$ a measurable map. Then T is ergodic if and only if for any bounded function $f : M \longrightarrow \mathbb{R}$, the function $\lim_n \frac{1}{n}\sum_{i=0}^{n-1}(T^*)^i f$ is constant a. e.

Proof: By von Neumann ergodic theorem, $\lim_n \frac{1}{n}\sum_{i=0}^{n-1}(T^*)^i f$ is T-invariant, hence constant a.e. whenever T is μ-ergodic. Conversely, if T is not μ-ergodic, there exists a bounded, measurable T-invariant function f which is not constant a.e., and $\lim_n \frac{1}{n}\sum_{i=0}^{n-1}(T^*)^i f = f$ is not constant. ∎

REMARK: Ergodicity would follow if $\lim_n \frac{1}{n}\sum_{i=0}^{n-1}(T^*)^i f = \text{const}$ for all $f = \chi_A$, where $A \subset M$ is a measurable subset, and χ_A its characteristic function.

COROLLARY: A dynamical system (M,μ,T) **is ergodic if and only if the eigenspace of the corresponding Koopman operator $T^* : L^2(M,\mu) \longrightarrow L^2(M,\mu)$ with eigenvalue 1 is 1-dimensional.**

Convergence in density (reminder)

DEFINITION: The **(asymptotic) density** of a subset $J \subset \mathbb{Z}^{\geqslant 1}$ is the limit $\lim_N \frac{|J \cap [1,N]|}{N}$. A subset $J \subset \mathbb{Z}^{\geqslant 1}$ has **density 1** if $\lim_N \frac{|J \cap [1,N]|}{N} = 1$.

DEFINITION: A sequence $\{a_i\}$ of real numbers **converges to a in density** if there exists a subset $J \subset \mathbb{Z}^{\geqslant 1}$ of density 1 such that $\lim_{i \in J} a_i = a$. The convergence in density is denoted by $\operatorname{Dlim}_i a_i = a$.

PROPOSITION: (Koopman-von Neumann, 1932) Let $\{a_i\}$ be a sequence of bounded non-negative numbers, $a_i \in [0, C]$. **Then convergence to 0 in density is equivalent to the convergence of Cesàro sums:**

$$\operatorname{Dlim}_i a_i = 0 \Leftrightarrow \lim_N \frac{1}{N} \sum_{i=1}^{N} a_i = 0$$

Mixing, weak mixing, ergodicity (reminder)

DEFINITION: Let (M, μ, T) be a dynamic system, with μ a probability measure. We say that

(i) T **is ergodic** if $\lim_n \frac{1}{n}\sum_{i=0}^{n-1} \mu(T^i(A) \cap B) = \mu(A)\mu(B)$, for all measurable sets $A, B \subset M$.

(ii) T **is weak mixing** if $\underset{i\to\infty}{\mathrm{Dlim}}\, \mu(T^i(A) \cap B) = \mu(A)\mu(B)$.

(iii) T **is mixing**, or **strongly mixing** if $\lim_{i\to\infty} \mu(T^i(A) \cap B) = \mu(A)\mu(B)$.

REMARK: **The first condition is equivalent to the usual definition of ergodicity by the previous remark.** Indeed, from (usual) ergodicity it follows that $\lim_n \frac{1}{n}\sum_{i=0}^{n-1}(T^*)^i(\chi_A) = \mu(A)$, which gives $\lim_n \frac{1}{n}\sum_{i=0}^{n-1}(T^*)^i(\chi_A)\chi_B = \mu(A)\chi(B)$ and the integral of this function is precisely $\mu(A)\mu(B)$. Conversely, if $\lim_n \int (T^*)^i(\chi_A)\chi_B$ depends only on the measure of B, the function $\lim_n \int (T^*)^i(\chi_A)$ is constant, hence T is ergodic in the usual sense.

REMARK: Clearly, (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) (the last implication follows because the density convergence implies the Cesàro convergence). ■

Mixing, weak mixing, ergodicity: spectral invariance (reminder)

Notice that the space generated by χ_A is C^0-dense in the space of all measurable functions. Therefore, in the definition of mixing/weak mixing/ergodicity we may replace χ_A, χ_B by arbitrary L^2-integrable functions. Denote by $\langle\cdot,\cdot\rangle$ the scalar product on the Hilbert space $L^2(M,\mu)$.

DEFINITION: Let (M,μ,T) be a dynamic system. We say that

(i) T **is ergodic** if $\lim_n \frac{1}{n}\sum_{i=0}^{n-1}\langle T^i(f),g\rangle\langle 1,1\rangle = \langle f,1\rangle\langle g,1\rangle$ for all $f,g\in L^2(M,\mu)$.

(ii) T **is weak mixing** if $\operatorname{Dlim}_{n\to\infty}\langle T^n(f),g\rangle\langle 1,1\rangle = \langle f,1\rangle\langle g,1\rangle$.

(iii) T **is mixing**, or **strongly mixing**, if $\lim_{n\to\infty}\langle T^n(f),g\rangle\langle 1,1\rangle = \langle f,1\rangle\langle g,1\rangle$.

REMARK: Notice that **these three notions are spectral invariant.** Indeed, the weakest of them already implies ergodicity, that is, the eigenspace of eigenvalue 1 of T is 1-dimensional. This implies that T determines the constant function in $L^2(M,\mu)$. However, the conditions (i)-(iii) are expressed in terms of 1, T and the scalar product, hence spectral invariant.

Mixing and weak mixing on the product

DEFINITION: Let (M, μ, T) be a dynamical system. Consider the dynamical system $(M, \mu, T)^2 := (M \times M, \mu \times \mu, T \times T)$, where $\mu \times \mu$ is the product measure on $M \times M$, and $T \times T(x, y) = (T(x), T(y))$.

THEOREM: Let (M, μ, T) be a dynamical system, and $(M, \mu, T)^2$ its product with itself. **Then $(M, \mu, T)^2$ is (weak) mixing if and only (M, μ, T) is (weak) mixing.**

Proof. Step 1: To simplify the notation, assume $\mu(M) = 1$. To see that (weak) mixing on $(M, \mu, T)^2$ implies the (weak) mixing on (M, μ, T), we take the sets $A_1 := A \times M$ and $B_1 := B \times M$. Then $\mu(T^i(A_1) \cap B_1) = \mu(T^i(A) \cap B)$ and $\mu(A_1)\mu(B_1) = \mu(A)\mu(B)$, hence

$$\lim_i \mu(T^i(A_1) \cap B_1) = \mu(A_1)\mu(B_1)$$

implies

$$\lim_i \mu(T^i(A) \cap B) = \mu(A)\mu(B).$$

Mixing and weak mixing on the product (2)

THEOREM: Let (M, μ, T) be a dynamical system, and $(M, \mu, T)^2$ its product with itself. **Then $(M, \mu, T)^2$ is (weak) mixing if and only (M, μ, T) is (weak) mixing.**

Step 2: Conversely, assume that (M, μ, T) is mixing. Since the subalgebra generated by cylindrical sets is dense in the algebra of measurable sets, it would suffice to show that $\lim_i \mu(T^i(A_1) \cap B_1) = \mu(A_1)\mu(B_1)$ where $A_1, B_1 \subset M^2$ are cylindrical. Write $A_1 = A \times A'$, $B_1 = B \times B'$. Then $\mu(T^n A_1 \cap B_1) = \Big(\mu(T^n A \cap B)\Big)\Big(\mu(T^n A' \cap B')\Big)$. The first of the terms in brackets converges to $\mu(A)\mu(B)$, the second to $\mu(A')\mu(B')$, giving

$$\lim_i \mu(T^i(A_1) \cap B_1) = \mu(A)\mu(B)\mu(A')\mu(B') = \mu(A_1)\mu(B_1).$$

■

REMARK: The same argument also proves that **ergodicity of $(M, \mu, T)^2$ implies ergodicity of (M, μ, T).** The converse implication is invalid even for a circle.

Ergodic measures which are not mixing

REMARK: Let $L_\alpha : S^1 \longrightarrow S^1$ be a rotation with irrational angle α. In angle coordinates on $S^1 \times S^1$, the rotation $L_\alpha \times L_\alpha$ acts as $L_\alpha \times L_\alpha(x, y) = (x+\alpha, y+\alpha)$. Therefore, the closure of the orbit of (x, y) is always contained in the closed set $\{(a, b) \in S^1 \times S^1 \ \mid \ a - b = x - y\}$, and $L_\alpha \times L_\alpha$ **has no dense orbits**.

This gives the claim.

CLAIM: Irrational rotation of a circle is ergodic, but not weakly mixing.

Proof: Otherwise, $L_\alpha \times L_\alpha$ would be weak mixing, and hence ergodic, on $S^1 \times S^1$. ■

Weak mixing and non-constant eigenfunctions

I am going to prove the following theorem.

Theorem 1: Let (M, μ, T) be a dynamical system. **Then the following are equivalent.**

(i) (M, μ, T) **is weakly mixing.**

(ii) The Koopman operator $T: L^2(M, \mu) \longrightarrow L^2(M, \mu)$ **has no non-constant eigenvectors.**

(iii) $(M, \mu, T)^2$ **is ergodic.**

The proof uses spectral theory of operators on a Hilbert space.

Tensor product of Hilbert spaces

DEFINITION: Let H, H' be two Hilbert spaces. The tensor product $H \otimes H'$ has a natural scalar product which is non-complete. Its completion $H \hat{\otimes} H'$ is called **completed tensor product** of H and H'.

REMARK: Let $\{e_i\}, \{e'_i\}$ be orthonormal bases in H, H'. **Then $H \hat{\otimes} H'$ is all series $\sum_i \alpha_{ij} e_i \otimes e'_j$ with $\sum_{i,j} |\alpha_{ij}|^2 < \infty$.**

REMARK: The natural map $H \hat{\otimes} H^* \stackrel{\Phi}{\longrightarrow} \operatorname{Hom}(H,H)$ is not surjective. Indeed, the identity operator $\sum_i e_i \otimes e_i^*$ does not belong to the completion of $H \otimes H^*$, because the series $1+1+1+1+\ldots$ does not converge.

CLAIM: Let (M,μ) and (M',μ') be metrizable spaces with Borel measure. **Then $L^2(M \times M', \mu \times \mu') = L^2(M,\mu) \hat{\otimes} L^2(M',\mu')$.**

Proof: The usual tensor product $C^0(M) \otimes C^0(M')$ is a dense (by Stone-Weierstrass) subring in $C^0(M \times M)$, the space $L^2(M,\mu) \otimes L^2(M',\mu')$ is its partial completion, and $L^2(M,\mu) \hat{\otimes} L^2(M',\mu')$ is its completion. Therefore, $L^2(M,\mu) \otimes L^2(M',\mu') \subset L^2(M,\mu) \hat{\otimes} L^2(M',\mu')$ is a dense subset. ■

Orthogonal operators on tensor square

Next lecture I will prove the following theorem.

THEOREM: Let U be an orthogonal operator on a Hilbert space H. **Then the following are equivalent:**

(i) U **has no eigenvectors in** H**.**

(ii) $U \times U$ **has no eigenvectors in** $H \hat{\otimes} H$ **with eigenvalue 1.**

This immediately implies equivalence of (ii) and (iii) in Theorem 1:

PROPOSITION: Let (M, μ, T) be a dynamical system. Then $T \times T$ **is ergodic on** M^2 **if and only if** T **has no non-constant eigenfunctions on** $L^2(M, \mu)$**.**

Proof: Let $H \subset L^2(M, \mu)$ be the space of all functions f with $\int_M f\mu = 0$. Then $L^2(M^2, \mu^2) = H \hat{\otimes} H \oplus H \oplus H \oplus \mathbb{R}$. Ergodicity of $T \times T$ on M^2 (and, hence, M) means that $T \times T$ has no invariant vectors in H and $H \otimes H$. **By the previous theorem, this is equivalent to** T **having no eigenvectors in** H**.** ■

Weak mixing and action on the square

Theorem 1: Let (M, μ, T) be a dynamical system. **Then the following are equivalent.**

(i) (M, μ, T) **is weakly mixing.**

(ii) The Koopman operator $T: L^2(M, \mu) \longrightarrow L^2(M, \mu)$ **has no non-constant eigenvectors.**

(iii) $(M, \mu, T)^2$ **is ergodic.**

Proof. Step 1: Equivalence of (iii) and (ii) is already proven. Implication (i) $\Rightarrow$ (iii) is elementary: indeed, $(M, \mu, T)^2$ is weakly mixing, hence ergodic. It remains only to prove that (iii) implies (i).

Weak mixing and action on the square (2)

Ergodicity of $(M,\mu,T)^2$ imlplies that (M,μ,T) is weak mixing:

Step 2: Let $A,B\subset M$ be measurable subsets. To simplify notation, we assume that $\mu(M)=1$. Consider the sequence $\frac{1}{n}\sum_{i=0}^{n-1}(\mu(T^iA\cap B)\mu(M)-\mu(A)\mu(B))^2$. The terms are non-negative, and by Koopman-von Neumann **convergence of this sequence implies density convergence of $\mu(T^iA\cap B)-\mu(A)\mu(B)$, which is the same as weak mixing.**

Step 3:

$$\frac{1}{n}\sum_{i=0}^{n-1}(\mu(T^iA\cap B)-\mu(A)\mu(B))^2=\left[\frac{1}{n}\sum_{i=0}^{n-1}\mu(T^iA\cap B)^2-\mu(A)^2\mu(B)^2\right]$$
$$+\left[\frac{2}{n}\sum_{i=0}^{n-1}\mu(A)^2\mu(B)^2-\mu(T^iA\cap B)\mu(A)\mu(B).\right]$$

The first term on RHS is $\frac{1}{n}\sum_{i=0}^{n-1}\mu((T\times T)^iA^2\cap B^2)-\mu(A^2)\mu(B^2)$, and it converges because $T\times T$ is ergodic. The second term is

$$-\mu(A)\mu(B)\frac{2}{n}\sum_{i=0}^{n-1}\mu(T^iA\cap B)-\mu(A)\mu(B),$$

and it converges because M is ergodic. ■