Teoria Ergódica Diferenciável

lecture 16: Tensor product and spectral theory

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Tensor product

DEFINITION: Let S be a set. Define **vector space, freely generated by S**, as the space of functions $\psi: \ S \longrightarrow k$ which are equal zero outside of a finite subset in S.

DEFINITION: Let V, V' be vector spaces over k, and W a vector space freely generated by $v \otimes v'$, with $v \in V, v' \in V'$, and $W_1 \subset W$ a subspace generated by combinations $av \otimes v' - v \otimes av'$, $a(v \otimes v') - (av) \otimes v'$, $(v_1 + v_2) \otimes v' - v_1 \otimes v' - v_2 \otimes v'$ and $v \otimes (v'_1 + v'_2) - v \otimes v'_1 - v \otimes v'_2$, where $a \in k$. Define **the tensor product** $V \otimes_k V'$ as a quotient vector space W/W_1.

PROPOSITION: ("Universal property of the tensor product")
For any vector spaces V, V', R, there is a natural identification $\operatorname{Hom}(V \otimes_k V', R) = \operatorname{Bil}(V \times V', R)$.

Proof: Clearly, any bilinear map $\rho \in \operatorname{Bil}(V \times V', R)$ defines a linear map $\tilde{\rho}: W \longrightarrow R$, and $\tilde{\rho}$ vanishes on W_1. This gives a map $\operatorname{Bil}(V \times V', R) \longrightarrow \operatorname{Hom}(V \otimes_k V', R)$. Inverse map takes $\tau \in \operatorname{Hom}(V \otimes_k V', R)$ and interprets it as a bilinear map in $\operatorname{Bil}(V \times V', R)$. ■

COROLLARY: For finite-dimensional V, V', one has $V \otimes_k V' = \operatorname{Bil}(V \times V', k)^*$.

Dimension of of tensor product

CLAIM: Dimension of $\operatorname{Bil}(V \times V', k)$ is equal to $\dim V \dim V'$.

Proof. Step 1: Let $\{\lambda_i\}$ be a basis in V^* and $\{\lambda'_i\}$ a basis in V'. Denote by $\{v_i\}$ $\{v'_i\}$ the dual basis in V, V'. Then $\lambda_i \lambda'_j$ can be interpreted as vectors in $\operatorname{Bil}(V \times V', k)$. These vectors are clearly linearly independent: indeed

$$\sum_{i,j} a_{ij} \lambda_i \lambda'_j (v_p, v'_q) = a_{pq}.$$

This gives $\dim \operatorname{Bil}(V \times V', k) \geqslant \dim V \dim V'$.

Step 2: On the other hand, $\dim V \otimes V' \leqslant \dim V \dim V'$, because it is generated by $v_p \otimes v_q$, hence $\dim \operatorname{Bil}(V \times V', k) \leqslant \dim V \dim V'$. ■

COROLLARY: Let $\{x_i\}$ and $\{y_i\}$ be bases in V, W. **Then $\{x_i \otimes y_j\}$ is a basis in $V \otimes_k W$.** ■

The space End(V)

Consider the space End(V) of endomorphisms of a vector space V (that is, of linear maps from V to itself). Given $x \in V, \lambda \in V^*$, consider the map $x \otimes \lambda \in$ End(V) mapping $y \in V$ to $x\lambda(y)$. This defines a bilinear map Bil($V \times V^*$, End(V)). As usual, we associate with this map a homomorphism $\Psi : V \otimes_k V^* \longrightarrow$ End(V).

THEOREM: The map $\Psi : V \otimes_k V^* \longrightarrow$ End(V) constructed above **is an isomorphism for any finite-dimensional space V.**

Proof: The dimensions of End(V) and $V \otimes V^*$ are equal to n^2, hence it suffices to show that Ψ is surjective. However, elements $x \otimes \lambda \in$ End(V) generate the space End(V) **(prove it).** ∎

Adjoint operators (reminder)

CLAIM: Let V be a Hilbert space, g a scalar product on V, and $A \in \operatorname{End}(V)$. **Then there exists a unique operator $A^* \in \operatorname{End}(V)$ such that $g(A(x), y) = g(x, A^*(y))$ for all $x, y \in V$.**

Proof: Let $x_1, ..., x_n, ...$ be an orthonormal basis in V, $A = (a_{ij})$ the matrix of A, and and A^t the transposed matrix $A^t = (a_{ji})$. Then $g(A(x_i), x_j) = a_{ij}$ and $g(x_i, A^*(x_j)) = a_ij$. This gives existence. Uniqueness is clear, because if $g(x, (A_1^* - A_2^*)(y)) = 0$ for all x, y, we have $A_1^* - A_2^* = 0$ **(prove it).** ■

DEFINITION: In this situation, the operator A^* is called **adjoint to A**. In orthonormal basis, **this operator is represented by the transposed matrix.**

CLAIM: $A \in O(V) \Leftrightarrow A^*A = 1$.

Proof: The equality

$$g(A(x), A(y)) = g(x, y) \qquad (a)$$

holds for all x, y if and only if

$$g(x, A^*A(y)) = g(x, y). \qquad (b)$$

■

Self-adjoint operators

DEFINITION: Let V be a vector space and $g \in \mathrm{Sym}^2 V$ a scalar product. An operator $A: V \longrightarrow V$ is called **self-adjoint** if $A = A^*$.

REMARK: In orthonormal basis, a self-adjoint operator is given by a matrix that satisfies $A = A^t$, that is, **symmetric**. The self-adjoint operators are often called **symmetric operators**.

Assume that V is finitely-dimensional.

CLAIM: Let A be a self-adjoint operator on (V, g), and $g_A(x, y) := g(A(x), y)$. **Then g_A is a bilinear symmetric form on V.** Moreover, **the map $A \mapsto g_A$ gives a bijective correspondence between self-adjoint operators and bilinear symmetric forms on V.**

Proof: Using g to identify V and V^*, we obtain that the spaces $V^* \otimes V^*$ of bilinear symmetric forms and $\mathrm{End}(V) = V \otimes V^*$ are also identified. This identification is given by a map $A \mapsto g(A(\cdot), \cdot)$. By definition, the form $g_A(\cdot, \cdot) := g(A(\cdot), \cdot)$. is symmetric if and only if A is self-adjoint. ■

REMARK: This is just another way to construct the well-known **bijective correspondence between symmetric matrices and bilinear symmetric forms.**

Normal form of a pair of bilinear symmetric forms

Theorem 1: (spectral theorem for self-adjoint operators) Let A be a self-adjoint operator on a finite-dimensional space (V, g). **Then A can be diagonalized in an orthonormal basis.**

Theorem $1'$: ("principal axis theorem") Let $V = \mathbb{R}^n$, and $h, h' \in \operatorname{Sym}^2 V^*$ be two bilinear symmetric forms, with h positive definite. **Then there exists a basis $x_1, ..., x_n$ which is orthonormal with respect to h, and orthogonal with respect to h'.**

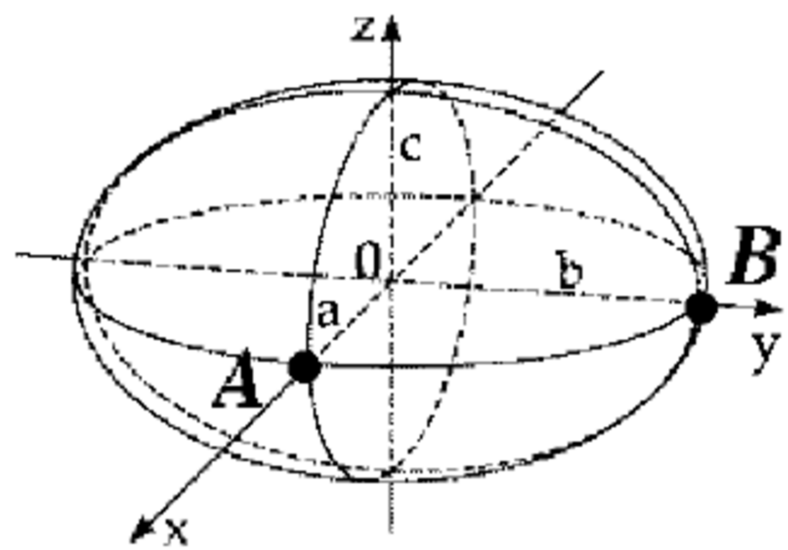
These theorems are clearly equivalent; I will give a proof later.

"principal axis theorem"

REMARK: Theorem $1'$ implies the following statement about ellipsoids: for any positive definite quadratic form q in $\mathbb{R}^n$, consider the ellipsoid

$$S = \{v \in V \ | \ q(v) = 1\}.$$

The group $SO(n)$ acts on $\mathbb{R}^n$ preserving the standard scalar product. **Then for some $g \in SO(n)$, $g(S)$ is given by equation $\sum a_i x_i^2 = 1$, where $a_i > 0$.** This is called **finding principal axes of an ellipsoid**.



Maximum of a quadratic form on a sphere

LEMMA: Let $V = \mathbb{R}^n$, and $h, h' \in \operatorname{Sym}^2 V^*$ be two bilinear symmetric forms, h positive definite, and $q(v) = h(v,v), q'(v) = h'(v,v)$ the corresponding quadratic forms. Consider q' as a function on a sphere $S = \{v \in V \mid q(v) = 1\}$, and let $x \in S$ be the point where q' attains maximum. Denote by $x^{\perp_h}$ and $x^{\perp_{h'}}$ the orthogonal complements with respect to h, h'. **Then $x^{\perp_h} = x^{\perp_{h'}}$.**

Proof: Since $q'(x)$ reaches maximum on a sphere, one has $\frac{d}{d\varepsilon} q'(x + \varepsilon v) = 2h'(x,v) = 0$ for any $v \in T_x S = x^{\perp_h}$. This gives $h'(x, x^{\perp_h}) = 0$. ■

Theorem $1'$: ("principal axis theorem") Let $V = \mathbb{R}^n$, and $h, h' \in \operatorname{Sym}^2 V^*$ be two bilinear symmetric forms, with h positive definite. **Then there exists a basis $x_1, ..., x_n$ which is orthonormal with respect to h, and orthogonal with respect to h'.**

Proof: Let $q(v) = h(v,v), q'(v) = h'(v,v)$ the corresponding quadratic forms. Consider q' as a function on a sphere $S = \{v \in V \mid q(v) = 1\}$, and let $x_1 \in S$ be the point where q' attains maximum. Then $x_1^{\perp_h} = x_1^{\perp_{h'}}$. Using induction, we may assume that on $x_1^{\perp_h}$, Theorem 1 is already proven, and there exists a basis $x_2, ..., x_n$ orthonormal for h and orthogonal for h'. **Then $x_1, x_2, ..., x_n$ is orthonormal for h and orthogonal for h'.** ■

Weak convergence

DEFINITION: Let $x_i \in H$ be a sequence of points in a Hilbert space H. We say that x_i **weakly converges** to $x \in H$ if for any $z \in H$ one has $\lim_i g(x_i, z) = g(x, z)$.

REMARK: Let $y(i) = \alpha_j(i) e_j$ be a sequence of points in a a Hilbert space with orthonormal basis e_i. **Then $y(i)$ converges to $y = \sum_j \alpha_j e_j$ if and only if $\lim_i \alpha_j(i) = \alpha_i$.**

CLAIM: For any sequence $\{y(i) = \sum_j \alpha_j(i) e_j\}$ of points in a unit ball, **there exists a subsequence $\{\tilde{y}(i) = \tilde{\alpha}_j(i) e_i\}$ weakly converging to $y \in H$.**

Proof: Indeed, $|\alpha_j(i)| \leqslant 1$, hence there exist a subsequence $\tilde{y}(i) = \tilde{\alpha}_j(i) x_j$ with $\tilde{\alpha}_j(i)$ converging for each j. The limit belongs to the unit ball because otherwise $\left|\sum_{j=1}^n \tilde{\alpha}_j(i) e_j\right| > 1$, which is impossible. ■

REMARK: Note that **the function $x \longrightarrow |x|$ is not continuous in weak topology.** Indeed, weak limit of $\{e_i\}$ is 0. The proof above shows that $|\cdot|$ is semicontinuous.

Compact operators

DEFINITION: Precompact set is a set which has compact closure. **A compact operator** is an operator which maps bounded sets to precompact.

EXAMPLE: Let $A \in \operatorname{Hom}(H, H)$ be an operator on Hilbert spaces and $\{e_i\}$ an orthonormal basis in H. Let $A(e_i) = z_i$; assume that $\sum |z_i|^2 < \infty$. **Then A is compact.**

Proof. Step 1: Let $y(i) = \alpha_j(i) e_j$ be a sequence of points in a unit ball. Replacing $y(i)$ by a subsequence, we may assume that $y(i)$ weakly converges to y.

Step 2: Then

$$\lim_i A(y(i)) = \lim_i \lim_n A\left(\sum_{j=1}^n \alpha_j(i)\right) = \lim_n \sum_{j=1}^n \alpha_j A(e_i)$$

and this sequence converges in the usual topology on H, because α_j are bounded and $\sum_i |A(e_i)|^2$ is bounded. ■

Compact operators and weak convergence

THEOREM: Let $A: H \longrightarrow H$ be a compact operator. **Then A maps any weakly convergent sequence to a convergent one.**

Proof: Let $\{y_i\}$ be a sequence which weakly converges to y. Replacing $\{y_i\}$ by a subsequence, we may assume that $A(y_i)$ converges to z. Then $\lim_i g(A(y_i), v) = g(z, v)$ for any $v \in H$. However,

$$\lim_i g(A(y_i), v) = \lim_i g(y_i, A^*(v)) = g(y, A^*(v)) = g(A(y), v).$$

Then $g(z, v) = g(A(y), v)$ for all $v \in H$, giving $z = A(y)$. ■

REMARK: Converse is also true: **you can characterize a compact operator as one which maps weakly convergent sequences to convergent.** Indeed, unit ball is weakly compact, as we have shown, hence its image is precompact for any map which takes the weakly convergent sequences to convergent.

Tensor product of Hilbert spaces (reminder)

DEFINITION: Let H, H' be two Hilbert spaces. The tensor product $H \otimes H'$ has a natural scalar product which is non-complete. Its completion $H \hat{\otimes} H'$ is called **completed tensor product** of H and H'.

REMARK: Let $\{e_i\}, \{e'_i\}$ be orthonormal bases in H, H'. **Then $H \hat{\otimes} H'$ is all series $\sum_i \alpha_{ij} e_i \otimes e'_j$ with $\sum_{i,j} |\alpha_{ij}|^2 < \infty$.**

REMARK: The natural map $H \hat{\otimes} H^* \stackrel{\Phi}{\longrightarrow} \operatorname{Hom}(H, H)$ is not surjective. Indeed, the identity operator $\sum_i e_i \otimes e_i^*$ does not belong to the completion of $H \otimes H^*$, because the series $1 + 1 + 1 + 1 + ...$ does not converge.

PROPOSITION: Let $\Phi \in H \hat{\otimes} H^*$, and $A: \ H \longrightarrow H$ be the corresponding operator. **Then A is compact.**

Proof: $\Phi = \alpha_{ij} e_i \otimes e_j$ with $\sum_{i,j} |\alpha_{ij}|^2 < \infty$, hence $A(e_i) = \sum_j \alpha_{ij} e_j$ satisfies $\sum_{i,j} |\alpha_{ij}|^2 < \infty$. Then it is compact as shown above. ■

Spectral theorem

THEOREM: (Spectral theorem for self-adjoint operators)
Let $A: \ H \longrightarrow H$ be a compact self-adjoint operator on a Hilbert space. **Then A can be diagonalized** in a certain orthonormal basis $e_1, e_2, ...$, with $\lim_i \alpha_i = 0$.

Proof. Step 1: The eigenvalues converge to 0 because A is compact. Let $B \subset H$ be the unit ball, and X the closure of $A(B)$. Denote by $x \in X$ the vector where $|x|$ is maximal. We shall prove that $x = A(z)$. To finish the proof of Spectral Theorem **it would suffice to show that z is an eigenvector and $A(z^\perp) \subset z^\perp$.**

Step 2: Let $z_i \in B$ be a sequence such that $\lim_i A(z_i) = x$. Replacing z_i by a subsequence, we may assume that z_i weakly converges to $z \in B$. Then $A(z) = x$, because A maps weakly convergent sequences to convergent. **This implies that $x \in \operatorname{im} A$.**

Step 3: Let $z \in H$ be an element of the unit sphere such that $A(z) = x$. Then $|A(z)| = \|A\|$. Since A is self-adjoint, $g(A(z), A(z)) = g(A^2(z), z) = \|A\|^2$. Since $g(A^2(z), z) = |z||A^2(z)| \cos\varphi$, where φ is an angle between x and $A(x)$, the equality $g(A^2(z), z) = |z||A^(z)|$ implies that z and $A^2(z)$ are proportional, hence x is an eigenvector for A^2.

Spectral theorem (2)

THEOREM: (Spectral theorem for self-adjoint operators) Let $A: H \longrightarrow H$ be a compact self-adjoint operator on a Hilbert space. **Then A can be diagonalized** in a certain orthonormal basis $e_1, e_2, ...$, with $\lim_i \alpha_i = 0$.

Steps 2-3: We have shown that there exists a vector $z \in H$ in a unit ball such that $|A(z)| = \|A\|$. Moreover, z is an eigenvector of A^2.

Step 4: Now, the function $q(z) = |A(z)|^2$ reaches its maximum on $z \in B$, hence $\frac{d}{d\varepsilon} q(z + \varepsilon v) = 2g(A(z), A(v)) = 0$ for all $v \in T_z S$, where $S \subset H$ is the unit sphere. This gives $z^\perp \supset \{v \in H \mid g(A(v), A(z)) = 0\}$. Since $g(A(v), A(z)) = g(v, A^2(z))$, we obtain $z^\perp \supset A^2(z^\perp)$. **We proved that A^2 is diagonal in an orthonormal basis.**

Step 5: This implies that H is an orthogonal direct sum of eigenspaces for A^2, which are finite-dimensional for non-zero eigenvalues, because A^2 is compact. Since A and A^2 commute, on each of these eigenspaces A acts as an adjoint operator, and we can apply the finite-dimensional spectral theorem. ■

Orthogonal operators on tensor square

THEOREM: Let U be an orthogonal operator on a Hilbert space H. **Then the following are equivalent:**

(i) U has no eigenvectors in H.

(ii) U (acting diagonally) has no eigenvectors in $H \hat{\otimes} H$ with eigenvalue 1.

Proof. Step 1: Implication (ii) $\Rightarrow$ (i) is clear. Indeed, a tensor square of a finite-dimensional space V with action of U contains a U-invariant vector corresponding to the Euclidean product $g \in \operatorname{Sym}^2(V^*) = \operatorname{Sym}^2(V) \subset V \otimes V$.

Step 2: The converse implication follows from the spectral theorem. Indeed, let $\Phi \in H \hat{\otimes} H$ be a U-invariant vector, and $A_1 : H \longrightarrow H$ be the corresponding U-invariant compact operator. Then $A := A_1^* A_1$ satisfies $g(A_1^* A_1 x, x) = g(A_1 x, A_1 x)$, hence it is a non-zero compact self-adjoint operator, which is diagonalizable with finite-dimensional eigenspaces. **Since U preserves these eigenspaces, it has non-zero eigenvectors.** ■