Teoria Ergódica Diferenciável

lecture 20: Expanding maps

Instituto Nacional de Matemática Pura e Aplicada

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Unique ergodicity (reminder)

DEFINITION: From now on in this lecture we consider dynamical systems (M, μ, T), where M is a compact space, μ a probability Borel measure, and $T: M \longrightarrow M$ continuous. We say that μ is **uniquely ergodic** if μ is a unique T-invariant probability measure on M.

REMARK: Clearly, **uniquely ergodic measures are ergodic.** Indeed, any T-invariant non-negative measurable function is constant a.e. in μ.

THEOREM: Let (M, μ, T) be as above, and μ uniquely ergodic. **Then the closure of any orbit of T contains the support of μ.**

THEOREM: Let (M, μ, T) be a dynamical system, with M a compact metric space. Denote by $C_n(f)$ the sum $\frac{1}{n}\sum_{i=0}^{n-1} T^i(f)$. **Then the following are equivalent.**

(i) (M, μ, T) is uniquely ergodic.

(ii) For any continuous function f, the sequence $C_n(f)$ converges everywhere to a constant.

(iii) For any continuous function f, the sequence $C_n(f)$ converges uniformly to a constant.

(iv) For any Lipschitz function f, the sequence $C_n(f)$ converges uniformly to a constant.

Riemannian manifolds (reminder)

DEFINITION: Let $h \in \mathrm{Sym}^2 T^*M$ be a symmetric 2-form on a manifold which satisfies $h(x,x) > 0$ for any non-zero tangent vector x. Then h is called **Riemannian metric**, of **Riemannian structure**, and (M,h) **Riemannian manifold**.

DEFINITION: For any $x, y \in M$, and any piecewise smooth path $\gamma : [a,b] \longrightarrow M$ connecting x and y, consider **the length** of γ defined as $L(\gamma) = \int_\gamma |\frac{d\gamma}{dt}| dt$, where $|\frac{d\gamma}{dt}| = h(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})^{1/2}$. Define **the geodesic distance** as $d(x,y) = \inf_\gamma L(\gamma)$, where infimum is taken for all paths connecting x and y.

EXERCISE: Prove that the **geodesic distance satisfies triangle inequality and defines a metric on M.**

EXERCISE: Prove that **this metric induces the standard topology on M.**

EXAMPLE: Let $M = \mathbb{R}^n$, $h = \sum_i dx_i^2$. **Prove that the geodesic distance coincides with $d(x,y) = |x - y|$.**

Covering maps

DEFINITION: Let $\varphi : \tilde{M} \longrightarrow M$ be a continuous map of manifolds (or CW complexes). We say that φ is **a covering** if φ is locally a homeomorphism, and for any $x \in M$ there exists a neighbourhood $U \ni x$ such that is a disconnected union of several manifolds U_i such that the restriction $\varphi\Big|_{U_i}$ is a homeomorphism.

THEOREM: A local homeomorphism of compacts spaces is a covering.

DEFINITION: Let Γ be a discrete group continuously acting on a topological space M. This action is called **properly discontinuous** if M is locally compact, and the space of orbits of Γ is Hausdorff.

THEOREM: Let Γ be a discrete group acting on a manifold (or CW-complex) M properly discontinuously. Suppose that the stabilizer group $\Gamma' : \operatorname{St}_\Gamma(x)$ is the same for all $x \in M$. **Then $M \longrightarrow M/\Gamma$ is a covering.** Moreover, **all covering maps are obtained like that.**

These results are left as exercises.

Finite coverings

EXAMPLE: A map $x \longrightarrow nx$ in a circle S^1 is a covering.

EXAMPLE: For any non-degenerate integer matric $A \in \operatorname{End}(\mathbb{Z}^n)$, the corresponding map of a torus T^n is a covering.

CLAIM: Let $\varphi: \tilde M \longrightarrow M$ be a covering, with M connected. **Then the number of preimages $|\varphi^{-1}(m)|$ is constant in M.**

Proof: Since $\varphi^{-1}(U)$ is a disconnected union of several copies of U, this number is a locally constant function of m. ■

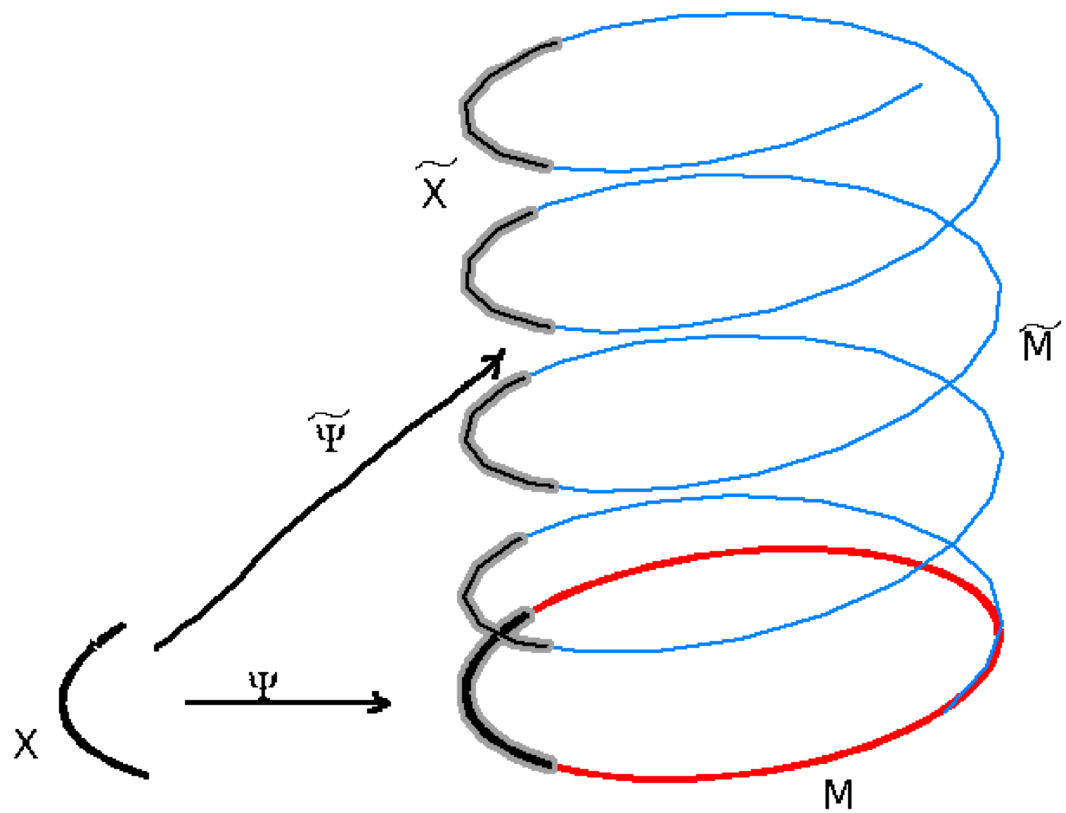
DEFINITION: Let $\varphi: \tilde M \longrightarrow M$ be a covering, with M connected. The number $|\varphi^{-1}(m)|$ is called **degree** of a map φ.

CLAIM: Any covering $\varphi: \tilde M \longrightarrow M$ with $\tilde M$ compact **has finite degree.**

Proof: Take U in such a way that $\varphi^{-1}(U)$ is a disconnected union of several copies of U, and let $x \in U$. Then $\varphi^{-1}(x)$ is discrete, and since $\tilde M$ is compact, any discrete subset of $\tilde M$ is finite. ■

Homotopy lifting

LEMMA: ("Homotopy lifting lemma") The map $\varphi: \tilde{M} \longrightarrow M$ is a covering iff φ is locally a homeomorphism, and for any path $\Psi: [0,1] \longrightarrow M$ and any $x \in \varphi^{-1}(\Psi(0))$, **there is a lifting $\tilde{\Psi}: [0,1] \longrightarrow \tilde{M}$ such that $\tilde{\Psi}(0) = x$ and $\varphi(\tilde{\Psi}(t)) = \Psi(t)$.**



Homotopy lifting

Expanding maps

DEFINITION: Let M be a compact Riemannian manifold. A smooth map $T: M \longrightarrow M$ is called **expanding** if there exists $A > 0$ and $\lambda > 1$ such that $|D(T^n)(v)| \geqslant A\lambda^n|v|$ for any tangent vector $v \in TM$.

REMARK: Any expanding map T is a local diffeomorphism, by inverse function theorem. Indeed, the differential $D(T^n)$ is everywhere invertible.

REMARK: By a result quoted above, this implies that T **is a finite covering.**

EXAMPLE: A map $x \longrightarrow nx$ in a circle S^1 is expanding.

EXAMPLE: For any non-degenerate integer matric $A \in \operatorname{End}(\mathbb{Z}^n)$, the corresponding map of a torus T^n is a covering. If, in addition, $|A(x)| > \operatorname{const}|x|$ for all $x \in \mathbb{R}^n$, it is expanding.

Expanding maps: independence from the metric

CLAIM: For any two Riemannian metrics g and g' on a compact manifold, **there exists a constant $C > 1$ such that for all $v \in TM$, $C^{-1}|v|_g \leqslant |v|_{g'} \leqslant C|v|_g$.**

Proof: Indeed, the function $|v|_g$ is continuous on the the compact space of $S_{g'}M = \{v \in TM \mid |v|_{g'} = 1\}$, and we can chose C such that $C^{-1} \leqslant |v|_g\Big|_{S_{g'}M} \leqslant C$. ■

REMARK: Let T be expanding on a Riemannian manifold (M, g'). Consider another Riemannian metric g. Then $|D(T^n)(v)|_g \geqslant C^{-1}|D(T^n)(v)|_{g'}$ and $|v|_g \leqslant C|v|_{g'}$. This gives

$$|D(T^n)(v)|_g \geqslant C^{-1}|D(T^n)(v)|_{g'} \geqslant C^{-1}A\lambda^n|v|_{g'} \geqslant C^{-2}A\lambda^n|v|_g,$$

and T is expanding in g, too. Therefore, T is expanding in g if and only if it is expanding in g': **the notion is metric-independent.**

Expanding maps: main result

THEOREM: Let M be a compact manifold and $T: M \longrightarrow M$ an expanding map. **Then there exists a unique T-invariant measure μ on M,** hence μ is uniquely ergodic. Moreover, (M, μ, T) **is mixing.**

This theorem will be proven later today.

REMARK: A T-invariant measure is often called **SRB (Sinai-Ruelle-Bowen) measure**

REMARK: If T is C^1, support of μ can be a very bad fractal set, but if it is C^2, there is a constant C such that $C^{-1}\,\mathrm{Vol} \leqslant \mu \leqslant C\,\mathrm{Vol}$, where Vol denotes the Riemannian volume measure.

Pushforward and pullback

DEFINITION: Let $T: M \longrightarrow M$ be a covering of degree q, and f a function on M. Define **pushforward** $T_* f$ as $T_*(x) = \frac{1}{q}\sum_{x_i \in T^{-1}(x)} f(x_i)$.

REMARK: Clearly, $T_* T^*(f) = f$.

DEFINITION: Given a measure μ, let $T^*\mu$ be a measure defined by $\int_M f T^*\mu := \int_M T_* f \mu$. This measure is called **pushforward of the measure μ**.

REMARK: The pushforward measure can be defined explicitly as follows. Let $U \subset M$ be an open subset such that $\varphi^{-1}(U)$ is a disconnected union of several copies of U, numbered as $U_1, ..., U_q$, and any $X \subset U$. Then $T^*\mu(X) = \frac{1}{q}\sum_{i=1}^q \mu(X_i)$, where $X_1, ..., X_q$ are preimages of X in $U_1, ..., U_q$.

Pushforward and pullback: strategy of the proof

REMARK: $\int_M fT^*\mu := \int_M T_*f\mu$ and $\int_M fT_*\mu := \int_M T^*f\mu$: **pullbacks and pushforwards are adjoint.** This is essentially a definition of pullback and pushforward for measures.

REMARK: Since $T_*T^*(f) = f$, this gives $\langle f, \mu\rangle = \langle T_*T^*f, \mu\rangle = \langle f, T^*T_*\mu\rangle$, where $\langle f, \mu\rangle = \int_M f\mu$ is the duality between measures and functions. **This gives $\mu = T^*T_*\mu$ for any measure μ on M.**

REMARK: Any T_*-invariant measure μ is also T^*-invariant, because $\mu = T^*T_*\mu = T^*\mu$.

REMARK: A priori, a T^*-invariant measure is not necessarily T_*-invariant. We will prove that for expanding maps **the T^*-invariant measure is unique,** By the previous remark, **any T_*-invariant measure is T^*-invariant, hence the T_*-invariant measure is also unique.**

Inverse to an expanding map

REMARK: An inverse to an expanding maps is a multivalued function **which is contracting on each branch**.

Let's state this more formally.

CLAIM 1: Let $T: M \longrightarrow M$ be an expanding map, $|D(T^n)(v)| \geqslant A\lambda^n|v|$ and $x, y \in M$. Then **for any preimage $\tilde{x} \in T^{-n}(x)$, there exists $\tilde{y} \in T^{-n}(y)$, such that $d(\tilde{x}, \tilde{y}) \leqslant \frac{d(x,y)}{A\lambda^n}$.**

Proof: Let $\gamma: [a,b] \longrightarrow M$ be a geodesic of length $d(x,y)$ connecting x to y. Using homotopy lifting, we lift γ to a map $\tilde{\gamma}: [a,b] \longrightarrow M$, with $T^n(\tilde{\gamma}) = \gamma$. Since $L_{\tilde{\gamma}} \leqslant A\lambda^n L_\gamma$, this gives $d(\tilde{x}, \tilde{y}) \leqslant \frac{d(x,y)}{A\lambda^n}$, where $\tilde{y} = \gamma(b)$. ■

COROLLARY: For any C-Lipschitz function f on M, $T^n_*(f)$ is $(A\lambda^n)^{-1}C$-Lipschitz.

Proof: Indeed,

$$|T^n_*(f)(x) - T^n_*(f)(y)| \leqslant q^{-n} \sum_{i=1}^{q^n} |f(\tilde{x}_i) - f(\tilde{y}_i)| \leqslant \frac{Cd(x,y)}{A\lambda^n},$$

where $\tilde{x}_i \in T^{-n}(x)$ are all preimages of x, and $\tilde{y}_i$ the preimages of y, associated with $\tilde{x}_i$ by homotopy lifting. ■

A T^*-invariant measure

DEFINITION: Diaeter of a metric space M is $\operatorname{diam}(M) := \inf_{x,y\in M} d(x,y)$.

COROLLARY: Let $T: M \longrightarrow M$ be an expanding map. **Then $T_*^n(f)$ converges uniformly to a constant.**

Proof: Since Lipschitz functions are C^0-dense in the space of continuous functions (Stone-Weierstrass), it suffices to prove the corollary when f is C-Lipschitz. Then it takes values in an interval I_0 of length δC, where $\delta := \operatorname{diam} C$. Since $T_*^n(f)$ is $(A\lambda^n)^{-1}C$-Lipschitz, $T^n(f)$ takes values in an interval I_n of length $(A\lambda^n)^{-1}C$. Then $I_0 \supset I_1 \supset ... \supset I_n \supset ...$ is a monotonous decreasing sequence of closed intervals, and their intersection is a single point $\mu(f) \in \mathbb{R}$ with the property $\sup_m |T_*^n(f) - \mu(f)| \leqslant (A\lambda^n)^{-1}C\delta$. ■

REMARK: My Riesz representation theorem, $f \longrightarrow \mu(f)$ defines a probabilistic measure on M. **Since $\mu(f) = \mu(T_*(f))$, this measure is T^*-invariant.**

CLAIM: A T^*-invariant probabilistic measure on M is unique.

Proof: Let ν be such a measure and f any Lipschitz function. Then $\int T_*^n(f)\nu = \int f\nu$, hence $\int f\nu = \lim_n \int T_*^n(f)\nu = \int \mu(f)\nu = \mu(f)$. ■

Unique ergodicity of T_*-invariant measure

COROLLARY: Let $T: M \longrightarrow M$ be an expanding map. **Then the T_*-invariant probability measure is unique** (and therefore, uniquely ergodic).

Proof: Let μ be a T_*-invariant measure; it exists by compactness of the measure space, as shown in Lecture 5. Since $T^*\mu = T^*T_*\mu = \mu$, this measure is T^*-invariant, but T^*-invariant measure is unique as shown above. ■

Volume functions (reminder)

Today I would repeat the content of the previous lecture, taking advantage of the material we have covered in September assignments.

DEFINITION: Let $\mathbf{C}$ be the set of compact subsets in a topological space M. A function $\lambda : \mathbf{C} \longrightarrow \mathbb{R}^{\geqslant 0}$ is

* **Monotone**, if $\lambda(A) \leqslant \lambda(B)$ for $A \subset B$
* **Additive**, if $\lambda(A \coprod B) = \lambda(A) + \lambda(B)$
* **Semiadditive**, if $\lambda(A \cup B) \leqslant \lambda(A) + \lambda(B)$

If these assumptions are satisfied, λ is called **volume function**.

DEFINITION: Let λ be a volume on M. For any $S \subset M$, define **inner measure** $\lambda_*(S) := \sup_C \lambda(C)$, where supremum is taken over all compact $C \subset S$, and **outer measure** $\lambda^*(S) := \inf_U \lambda_*(U)$, where infimum is taken over all open $U \supset S$.

THEOREM: (Carathéodory)
The outer measure is a measure on the Borel σ-algebra.

T_*-invariant volume function

Let $T: M \longrightarrow M$ be an expanding map of degree q. A T_*-invariant volume function is constructed as follows. Let $x \in M$ be a point. Consider the sets $S_0 = \{x\}$, $S_1 = T^{-1}(S_0), ..., S_n = T^{-1}(S_{n-1})$.

Given a compact $K \subset M$, let

$$\rho(K) := \overline{\lim}_n \frac{1}{q^n} |K \cap S_n|$$

Clearly, ρ is a T_*-invariant volume function, and $\rho(M) = 1$, hence **the corresponding outer measure is T_*-invariant and probabilistic.**

Mixing

CLAIM: Let (M, μ, T) be the expanding dynamical system, with T of degree q. Then $\int_M T_*(f) g\mu = \int f T^*(g)\mu$.

Proof: Clearly, $T_*(f)g(x) = \frac{1}{q}\sum_{x_i \in T^{-1}(x)} f(x_i)g(x)$, and $fT^*(g)(x) = f(x)g(T(x))$. Then $T^*(T_*(f)g) = \frac{1}{q}\sum_{x_i \in T^{-1}(x)} f(T(x_i))g(T(x)) = fT^*(g)(x)$. Since μ is T^*-invariant, this implies $\int_M T_*(f) g\mu = \int f T^*(g)\mu$. ■

COROLLARY: $\lim_n \int_M (T_*)^n(f) g\mu = \mu(f)\mu(g)$

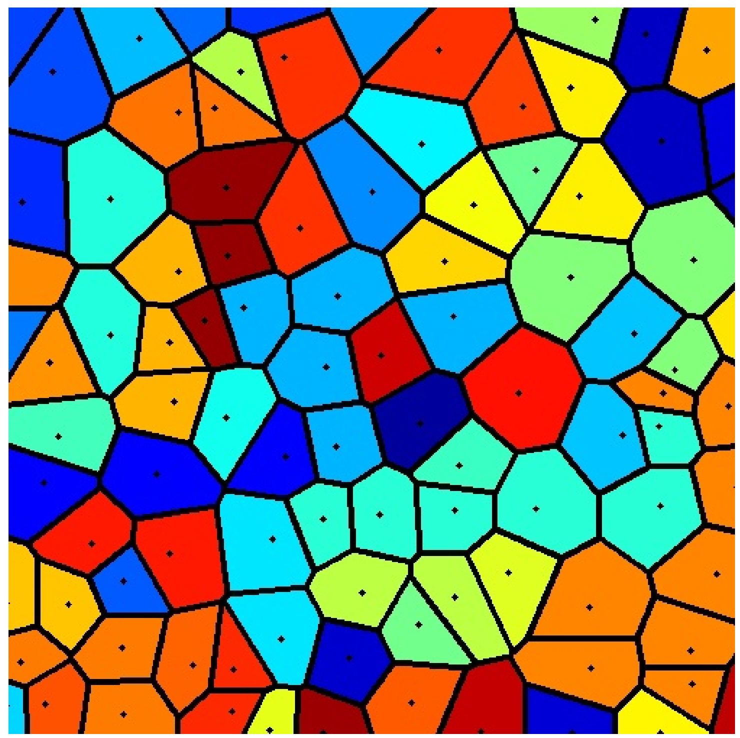
Proof: As shown above, $\int_M (T_*)^n(f) g\mu = \int_M f(T^*)^n g\mu$. Since $(T^*)^n g$ uniformly converges to $\mu(f)$, the integral $\int_M f(T^*)^n g\mu$ converges to $\mu(f)\int_M g\mu = \mu(f)\mu(g)$. ■

COROLLARY: An expanding dynamical system (M, μ, T) is mixing.

Proof: The relation $\lim_n \int_M (T_*)^n(f) g\mu = \mu(f)\mu(g)$ is one of the definitions of mixing systems. ■

Voronoi partitions

DEFINITION: Let M be a metric space, and $S \subset M$ a finite subset. **Voronoi cell** associated with $x_i \in S$ is $\{z \in M \ \ | \ (z, x_i) \leqslant d(z, x_i) \forall j \neq i\}$. **Voronoi partition** is partition of M onto its Voronoi cells.



Voronoi partition

Voronoi partitions and expanding maps

CLAIM: Let $T: M \longrightarrow M$ be an expanding map on a Riemannian manifold (M, g), $x \in M$ a point, and $S_0 = \{x\}$, $S_1 = T^{-1}(S_0), ..., S_n = T^{-1}(S_{n-1})$. Denote by g_i the Riemannian metric $(T^i)^*(g)$, and let $\mathcal{V}_i$ be the Voronoi partition of (M, g_i) associated with S_i. **Then for each cell P of $\mathcal{V}_i$. the set $T(P)$ is a Voronoi cell of $\mathcal{V}_{i-1}$.**

Proof: The map $T: (M, g_i) \longrightarrow (M, g_{i-1})$ is a local isometry mapping centers of Voronoi partition $\mathcal{V}_i$ to centers of $\mathcal{V}_{i-1}$. ■

REMARK: For each Voronoi sell P in $\mathcal{V}_n$, one has $T_n(P) = M$. Then $\rho(P) \geqslant \frac{1}{q^n}$. Therefore, **a set which contains a Voronoi cell has positive measure.**

REMARK: Let (M, μ, T) be an expanding system. As indicated above, to show that M is support of μ, it would suffice to show that each open set contains a Voronoi cell of $\mathcal{V}_n$, for n sufficiently big.