Teoria Ergódica Diferenciável

lecture 22: Ratner theory

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Representing numbers by quadratic forms

DEFINITION: Let q be a quadratic form on $\mathbb{R}^n$. We say that α **is represented by** q if $q(v) = \lambda$ for some $v \in \mathbb{Z}^n$.

THEOREM: (Lagrange)
Any positive integer is represented by the form $x^2 + y^2 + z^2 + t^2$.

THEOREM: (290-theorem; Bhargava, Hanke)
Let q be a quadratic form with integer coefficients, representing 1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, 290. **Then q represents all positive integers.**

DEFINITION: A quadratic form q is **irrational** if q is not proportional to a form with rational coefficients.

THEOREM: (Oppenheim conjecture, 1929; proven by G. Margulis, 1987)
Let q be an irrational quadratic form on $\mathbb{R}^{n+m}$, $n, m > 0$, $n + m > 2$, and S the set of numbers represented by q. **Then S is dense in $\mathbb{R}$.**

The proof of this result **is based on ergodic theory.**

Haar measure

DEFINITION: (Left) Haar measure on a locally compact topological group G is a left-invariant, locally finite Borel measure.

THEOREM: Haar measure exists on each locally compact topological group, and is unique up to a constant multiplier.

REMARK: In Lecture, 13, **we have seen this for Lie groups** and measures associated with differential forms.

REMARK: Since the left action on a group commutes with the right action, right translations map any left-invariant measure to a left-invariant. This means that the **left Haar measure is multiplied by a constant under a right translation.**

Unimodular groups

DEFINITION: A group is called **unimodular** if the left Haar measure is right-invariant.

REMARK: In other words, **a group is unimodular if the left Haar measure is equal to the right Haar measure.**

EXAMPLE: The group of affine transforms on $\mathbb{R}$ or on $\mathbb{R}^n$ **is not unimodular** (prove it)

DEFINITION: A character of a group is a homomorphism to the multiplicative group $\mathbb{C}^*$ or $\mathbb{R}^*$ or $\mathbb{Q}^*$.

REMARK: Since **the right action multiplies the Haar measure by a character χ, any group G which satisfies $G = [G, G]$ is unimodular.**

Lattices in Lie groups

DEFINITION: Let $\Gamma \subset G$ be a discrete subgroup in a Lie group, and $\pi : G \longrightarrow G/\Gamma$ the corresponding covering map (we take the quotient G/Γ with respect to the left action). Since π is locally a diffeomorphism, and Γ preserves the measure, there is a measure μ on G/Γ such that for all $U \subset G$ with $\pi : U \longrightarrow \pi(U)$ a diffeomorphism, the restriction $\pi|_U$ preserves the measure. This measure is called **Haar measure on G/Γ.**

DEFINITION: A discrete subgroup $\Gamma \subset G$ is called **a lattice** if the Haar measure of G/Γ is finite.

CLAIM: Let G be a Lie group which contains a lattice Γ. **Then G is unimodular.**

Proof: Consider the right action R^g of G on G/Γ. Then $R^g_*(\mu) = \chi(g)\mu$, where μ denotes the Haar measure. However, the volume of G/Γ has to stay constant, because R^g is a diffeomorphism. This gives

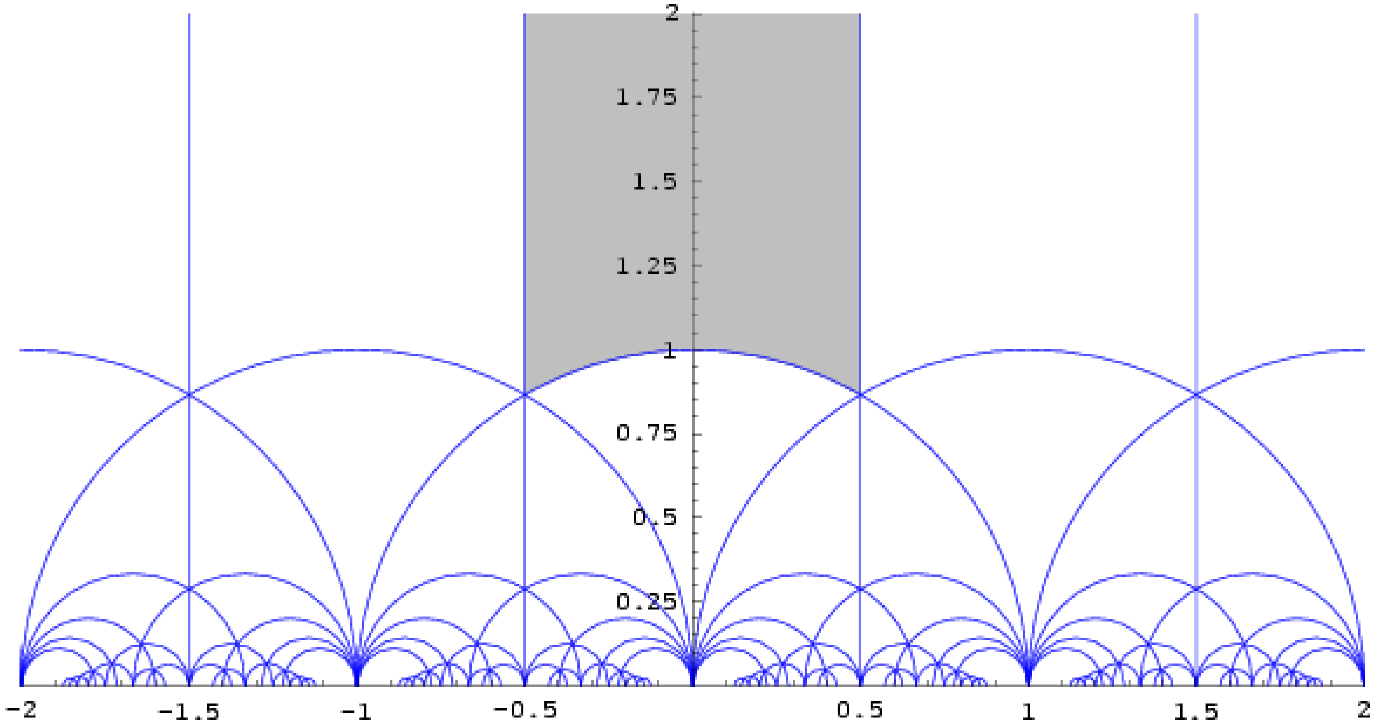
$$\int_{G/\Gamma} \mu = \int_{G/\Gamma} R^g_*(\mu) = \int_{G/\Gamma} \chi(g)\mu = \chi(g) \int_{G/\Gamma} \mu$$

and $\chi(g) = 1$. ■

Fundamental domain

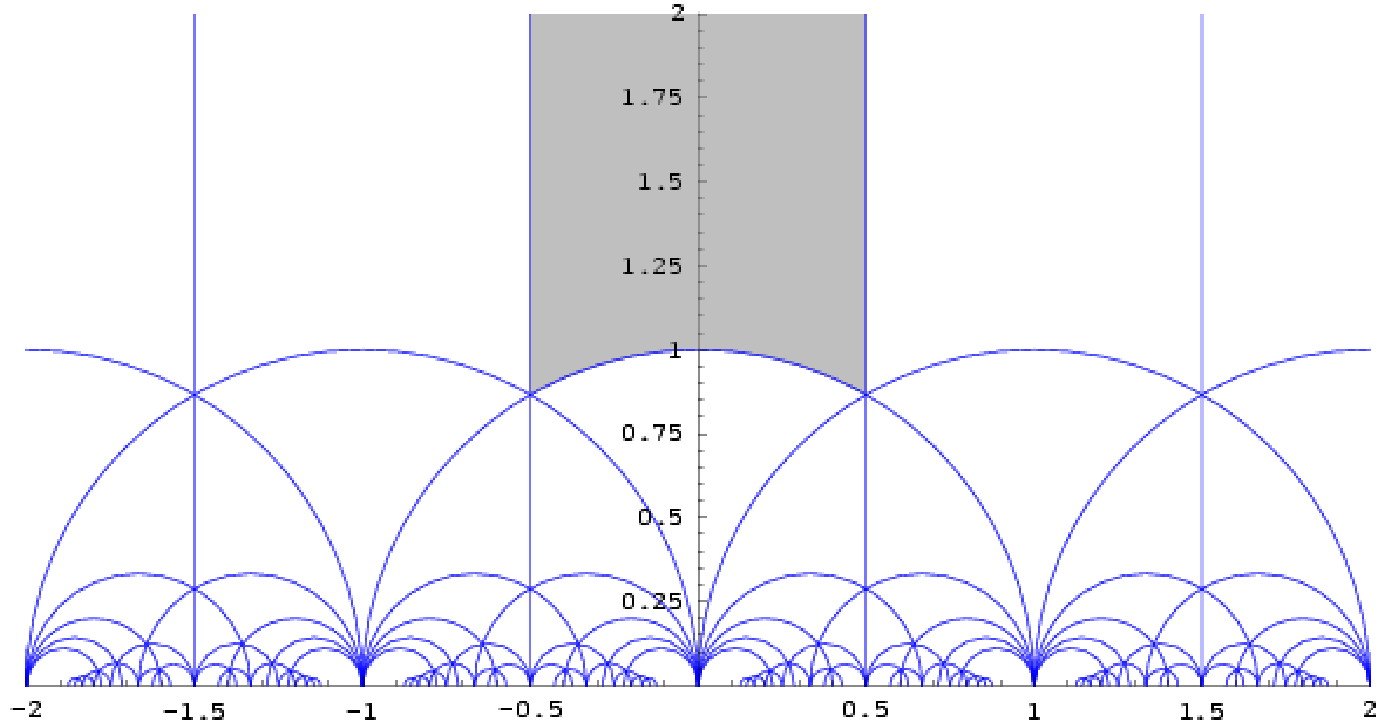
DEFINITION: Let Γ be a discrete group acting on a space M with measure properly discontinuously. The **fundamental domain** of this action is a subset $D \subset M$ intersecting each orbit of Γ exacty once outside of measure 0.

REMARK: Clearly, **a subgroup $\Gamma \subset G$ is a lattice $\Leftrightarrow$ its fundamental domain in G has finite volume.**



Fundamental domain of $SL(2,\mathbb{Z})$ acting in the Poncaré upper half-plane.

Fundamental domain (2)



Fundamental domain of $SL(2,\mathbb{Z})$ acting in the Poncaré upper half-plane.

REMARK: From this picture it is easy to see that $SL(2,\mathbb{Z})$ **is a lattice in $SL(2,\mathbb{R})$.** Indeed, the fundamental domain Ω of $\Gamma := SL(2,\mathbb{Z})$ acting in Poincare plane $\mathbb{H}^2 = SL(2,\mathbb{R})/S^1$ has finite volume, because it is a triangle. This implies that the fundamental domain of Γ in $SL(2,\mathbb{R})$, which is fibered over Ω with compact fiber S^1, **also has finite volume.**

Borel and Harish-Chandra theorem

DEFINITION: An **algebraic group** is a subgroup $G \subset GL(n)$ defined by polynomial equations.

REMARK: In fact, **any connected Lie sugbroup $G \subset GL(n, \mathbb{R})$ is a connected component of an algebraic group.** Moreover, **any complex Lie subgroup of $GL(n, \mathbb{C})$ is algebraic.**

DEFINITION: **A rational algerbraic group** $G \subset GL(n, \mathbb{R})$ is a Lie subgroup of $GL(n, \mathbb{R})$ defined by polynomial equations with rational coefficients. We denote by $G_{\mathbb{Z}}$ (or $G_{\mathbb{Q}}$) the subgroup of G consisting of all integer (rational) matrices. **A rational character** on G is a group homomorphism $G_{\mathbb{Q}} \longrightarrow \mathbb{Q}^{>0}$ into multiplicative group of positive rational numbers.

THEOREM: **(Borel and Harish-Chandra)**
Let $G \subset GL(n, \mathbb{R})$ be a rational algebraic group which has no non-trivial rational characters. **Then $G_{\mathbb{Z}} = G \cap SL(n, \mathbb{Z})$ is a lattice on G.**

REMARK: This is a non-trivial theorem, but we have proved it for $G = SL(2, \mathbb{R})$ already. **It can be easily proven for (some) other groups by constructing the fundamental domain explicitly.**

Jordan-Chevalley decomposition

DEFINITION: A matrix $g \in GL(n)$ is called **unipotent** if all its eigenvalues are equal 1, and **semisimple** if it is diagonalizable over $\mathbb{C}$.

THEOREM: Let $G \subset GL(n)$ be an algerbaic group. Then **any $g \in G$ has a decomposition $g = su$, where $s \in G$ is semisimple, $u \in G$ is unipotent, and s, u commute.** Moreover, **such decomposition is unique and functorial under algebraic group homomorphisms.**

DEFINITION: This decomposition is called **the Jordan-Chevalley decomposition**.

REMARK: For $G = GL(n)$, **Jordan-Chevalley decomposition is the same as the usual Jordan normal form.**

Groups generated by unipotents

DEFINITION: We say that an algebraic group G is **generated by unipotents** if any element of G can be represented a product of unipotent elements.

REMARK: For each unipotent $u \in G$, and each $g \in G$, the element gug^{-1} is also unipotent (indeed, both are exponents of a nilpotent matrix). Therefore, **the subgroup $G' \subset G$ generated by unipotents is normal.**

COROLLARY: Let G be a simple algebraic group (such as $SL(n), SO(n), Sp(n), ...$) containing a unipotent element. **Then G is generated by unipotents.**

Proof: Since the map $x \longrightarrow gxg^{-1}$ preserves unipotents, the subgroup $H \subset G$ generated by unipotents is normal. Then $H = G$ because G is simple. ∎.

EXAMPLE: A compact Lie group has no non-trivial unipotents, because each element of a compact group is semisimple.

C. Moore theorem

THEOREM: (C. Moore, 1966) Let $\Gamma \subset G$ be lattice in a simple algebraic group, and $H \subset G$ a non-compact subgroup. **Then the action of H on G/Γ is ergodic.**

REMARK: This implies, in particular, that **general orbits of H-action on G/Γ, or of Γ-action on G/H, are dense.**

COROLLARY: The group $SL(n,\mathbb{Z})$ acts on $SL(n,\mathbb{R})/H$ with dense orbits, for any non-compact Lie subgroup $H \subset SL(n,\mathbb{R})$.

Ergodic decomposition for G/Γ

Let $\Gamma \subset G$ be lattice in an algebraic group and $H \subset G$ a subgroup generated by unipotents. Ratner measure classification theorem classifies the H-ergodic measures on G/Γ.

DEFINITION: Let $\Gamma \subset G$ be lattice in an algebraic group and $S \subset G$ a subgroup such that $S \cap \Gamma$ is a lattice in S. Denote by $\tilde{\mu}_S$ the Haar measure from $S/\Gamma \cap S$, and let $S/\Gamma \cap S \xrightarrow{j} G/\Gamma$ be a natural embedding. **An algebraic measure** on G/Γ is $\mu_S := L^g_* j_* \tilde{\mu}_S$, where $g \in G$ and L^g is the left action of g.

THEOREM: (Ratner theorem on measure classification) Let $\Gamma \subset G$ be a lattice in an algebraic group and $H \subset G$ a subgroup generated by unipotents. Consider the minimal subgroup $S \subset G$ such that $S \cap \Gamma$ is a lattice in S, containing $x^{-1}Hx$ for some $x \in G$. **Then $\mu_S = L^x_* j_* \tilde{\mu}_S$ is an H-ergodic measure. Moreover, all H-ergodic measures are obtained this way.**

Ratner theorem on classification of orbits

THEOREM: Let $H \subset G$ be a Lie subroup generated by unipotents, and $\Gamma \subset G$ an arithmetic lattice. Then **the closure of any Γ-orbit in G/H is an orbit of a Lie subgroup $S \subset G$, such that $S \cap \Gamma \subset S$ is a lattice.**

COROLLARY: Let q be a quadratic form on $\mathbb{R}^{m+n}$, of signature (m, n), $m, n > 0$, $m + n > 2$, $G = SL(n + m)$, $\Gamma = SL(m + n, \mathbb{Z})$, and $H = SO(q) \subset G$. Then an orbit $H \cdot e$ **is dense in G/Γ for irrational q and closed for rational q.**

Proof. Step 1: Ratner theorem implies that $\overline{H \cdot x} = Sx$, where S is a minimal Lie group containing xHx^{-1} and such that $xSx^{-1} \cap \Gamma$ is a lattice.

Step 2: It is not hard to see that if $S \cap \Gamma$ is a lattice, then S is rational.

Step 3: Any connected Lie subgroup of $SL(n, \mathbb{R})$ containing H is equal to H or to G: also not hard to check. Therefore, **closed orbit of H correspond to rational subgroups $xHx^{-1} \subset G$, and non-closed are dense.**

Step 4: For q irrational, $SO(q)$ is not rational, and $H \cap \Gamma$ is not a lattice. ■

Proof of Oppenheim conjecture

THEOREM: (Oppenheim conjecture)
Let q be an irrational quadratic form on $\mathbb{R}^{n+m}$ if signature $n, m > 0$, $n+m > 2$, and S the set of numbers represented by q. **Then S is dense in $\mathbb{R}$.**

Proof. Step 1: Let $G = SL(n+m)$, and $H = SO^+(q) \subset G$. By the previous Corollary, left orbit of $H \cdot e$ is dense in G/Γ. Clearly, this is equivalent to the right orbit $e \cdot \Gamma$ being dense in the left quotient $H\backslash G$.

Step 2: Consider the function $Q_{e_0} : H\backslash G \longrightarrow \mathbb{R}$ mapping g to $q(g(e_0))$, where $e_0 = (1, 0, 0, ..., 0)$. **Then $Q_{e_0}(e \cdot \Gamma)$ is the set of all numbers represented by q.**

Step 3: Since $e \cdot \Gamma$ is dense in $H\backslash G$, the image $Q_{e_0}(e \cdot \Gamma)$ is dense in $\mathbb{R}$. ■

Marina Ratner (1938-2017)

Marina Ratner (1984).