Hyperkähler manifolds 13: Verma modules

Definition 13.1. Let $\mathfrak{g}$ be a Lie algebra. **The universal enveloping algebra** $U_{\mathfrak{g}}$ is the quotient of the free associative algebra $\bigoplus_{i=0}^{\infty} \mathfrak{g}^{\otimes i}$ generated by $\mathfrak{g}$ by the ideal geberated by all relations $xy-yx = [x,y]$, where xy denotes the multiplicative operation on $U_{\mathfrak{g}}$, and $[x,y]$ the commutator in $\mathfrak{g}$

Exercise 13.1. Prove that the universal enveloping algebra $U_{\mathfrak{sl}(2)}$ is infinitely-dimensional.

Exercise 13.2. Let $g \in \mathfrak{g}$.

a. Prove that the operation $g(\phi) = g\phi$ defines a structure of $\mathfrak{g}$-representation on $U_{\mathfrak{g}}$.

b. Prove that any irreducible representation of $\mathfrak{g}$ is isomorphic to a quotient of $U_{\mathfrak{g}}$.

Exercise 13.3. Let $\mathfrak{g}$ be a Lie algebra. Prove that the catefory of $U_{\mathfrak{g}}$-representations is naturally equivalent to the category of representations of $\mathfrak{g}$.

Definition 13.2. Let $\mathfrak{b}^+ \subset \mathfrak{sl}(2)$ be the upper triangular Borel subalgebra, that is, the 2-dimensional algebra of upper triangular matrices, and I_p its 1-dimensional representation, where the nilpotent element acts as 0 and the diagonal element $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ acts as a multiplication by p. **The Verma module of highest weight** p for $\mathfrak{sl}(2)$ is $I_p \otimes_{U_{\mathfrak{b}^+}} U_{\mathfrak{sl}(2)}$.

Exercise 13.4. Prove that the Verma module is generated (as an $\mathfrak{sl}(2)$-module) by one vector.

Exercise 13.5 (*). Prove that any Verma module is infinite-dimensional.

Exercise 13.6. a. Prove that any finite-dimensional irreducible representation of $\mathfrak{sl}(2)$ is a quotient of a Verma module.

b. (*) Construct an irreducible representation of $\mathfrak{sl}(2)$ which is not a quotient of a Verma module.

Definition 13.3. We say that a representation of $\mathfrak{sl}(2)$ **has a highest vector** if it contains a vector on which the nilpotent part of $\mathfrak{b}^+$ acts trivially. and the diagonal part $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ acts as a multiplication by a constant.

Exercise 13.7. Prove that any finite-dimensional representation of $\mathfrak{sl}(2)$ contains a highest vector.

Exercise 13.8 (!). Prove that any Verma module has precisely one irreducible quotient. Prove that this defines a bijection between the set of Verma modules and the set of isomorphism classes of irreducible representations containing a highest vector.

Exercise 13.9. Let V be a finite-dimensional irreducible representation of $\mathfrak{sl}(2)$.

a. Prove that the highest vector of V is unique up to a constant.

b. Let $\mathfrak{V}_p$ be a Verma module which admits a surjective map $\phi: \mathfrak{V}_p \longrightarrow V$. Prove that ϕ is unique up to a constant, and maps a highest vector to a highest vector.

Hint. Use the previous exercise.

Exercise 13.10. a. Prove that a finite-dimensional representation of $\mathfrak{sl}(2)$ is uniquely determined by the weight p of its highest weight vector.

b. Prove that p is integer and non-negative.

Hint. Use the previous exercise.

Exercise 13.11 (!). Prove that any finite-dimensional representation V of $\mathfrak{sl}(2)$ is isomorphic to a symmetric power of the fundamental representation.

Hint. Use the previous exercise.

Exercise 13.12 (*). Prove that the Verma module is irreducible for any $p \notin \mathbb{Z}$.

Exercise 13.13 (*). Prove that the Verma module is irreducible for any $p < 0$.

Exercise 13.14 (*). Consider the Lie algebra $\mathfrak{su}(2) = \langle I, J, K \rangle$ of imaginary quaternions. The element $B := I^2 + J^2 + K^2 \in U_{\mathfrak{su}(2)}$ is called **the Casimir element**. Prove that the Casimir element is central, and all central elements of $U_{\mathfrak{su}(2)}$ are polynomials in B