Hyperkähler manifolds 14: Hodge structures

In this handout you may freely use the standard facts of Hodge theory on compact Kähler manifolds, except where indicated otherwise; however, most exercises have solutions which are independent from the Hodge theory.

14.1 Hodge structures of weight 1

Definition 14.1. Let $V_{\mathbb R}$ be a real vector space. **A (real) Hodge structure of weight** w on a vector space $V_{\mathbb C} = V_{\mathbb R}\otimes_{\mathbb R}{\mathbb C}$ is a decomposition $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$, satisfying $\overline{V^{p,q}} = V^{q,p}$. It is called **rational** or **integer** Hodge structure if we fix a rational lattice $V_{\mathbb Q}\subset V_{\mathbb R}$ or an integer lattice $V_{\mathbb Z}\subset V_{\mathbb R}$ such that $V_{\mathbb R} = V_{\mathbb Q}\otimes_{\mathbb Q}{\mathbb R}$ or $V_{\mathbb R} = V_{\mathbb Z}\otimes_{\mathbb Z}{\mathbb R}$. A Hodge structure is equipped with $U(1)$-action, with $u\in U(1)$ acting as u^{p-q} on $V^{p,q}$. **Morphism** of Hodge structures is a map $\rho:\ V_{\mathbb R}\longrightarrow W_{\mathbb R}$ which is $U(1)$-invariant; if V, W are rational (integer) Hodge structures, we ask ρ to be rational (integral). Morphisms of Hodge structures are also called **the Hodge morphisms**.

Remark 14.1. It is convenient to assume that $V^{p,q}=0$, unless $p,q\geqslant 0$. In this handout we will follow this convention (which is not universal).

Exercise 14.1. Prove that the Hodge structure of weight 1 on $V_{\mathbb R}$ is uniquely defined by an operator $I\in \operatorname{End}(V_{\mathbb R})$, $I^2=-\mathsf{Id}$, which satisfies $I\big|_{V^{1,0}}=\sqrt{-1}$ and $I\big|_{V^{0,1}}=\sqrt{-1}$. Prove that a map $V_{\mathbb R}\longrightarrow W_{\mathbb R}$ is a morphism of Hodge structures if and only if it commutes with I.

Exercise 14.2. Let $\Lambda={\mathbb Z}^{2n}\subset V={\mathbb C}^n$ be a discrete integer lattice. Prove that any Λ-invariant holomorphic function on V is constant.

Exercise 14.3 (!). A **complex torus** is a quotient ${\mathbb C}^n/\Lambda$, where $\Lambda={\mathbb Z}^{2n}$ is a discrete, cocompact integer lattice. Prove that any holomorphic map $A\longrightarrow B$ of complex tori $M_1=V_1/\Lambda_1$ $M_2=V_2/\Lambda_2$ is induced by a linear map $V_1\longrightarrow V_2$ taking Λ_1 to Λ_2.

Hint. Use the previous exercise.

Exercise 14.4. Let M be a complex torus.

a. (!) Let $H^0(M,\Omega^1 M)$ be the space of holomorphic 1-forms. Prove that all forms $\xi\in H^0(M,\Omega^1 M)$ are closed (without using the Hodge theory),

b. (!) Prove that the natural map $H^0(M,\Omega^1 M)\longrightarrow H^1(M,{\mathbb C})$ is injective. We denote its image by $H^{1,0}(M)$.

c. Prove that $H^1(M,{\mathbb C}) = H^{1,0}(M)\oplus\overline{H^{1,0}(M)}$.

d. Prove that $V_{\mathbb Z}:=H^1(M,{\mathbb Z})$ is equipped with a structure of an integer Hodge structure such that $V^{1,0}=H^{1,0}(M)$

e. (!) Prove that any morphism of weight 1 Hodge structures $\rho : W \longrightarrow V$ associated with complex tori B, A is induced by a holomorphic map $A \longrightarrow B$.

Definition 14.2. Let M_1, M_2 be compact complex tori. **An isogeny** is a finite covering $M_1 \longrightarrow M_2$.

Exercise 14.5. Let M_1, M_2 be compact complex tori. Prove that isogenies $M_1 \longrightarrow M_2$ are in bijective correspondence with integral bijective morphisms of the corresponding integer Hodge structures.

Exercise 14.6. An exact sequence $0 \longrightarrow U \longrightarrow W \longrightarrow V \longrightarrow 0$ of rational Hodge structures is **split** if there exists a rational Hodge substructure $V_1 \subset W$ which is projected to V bijectively. Let $A \longrightarrow B \stackrel{\phi}{\longrightarrow} C$ be a sequence of holomorphic maps of complex tori, such that the corresponding sequence of rational Hodge structures $0 \longrightarrow U \longrightarrow W \longrightarrow V \longrightarrow 0$ is exact. Prove that it is split if and only if there exists a holomorphic map $\psi : C \longrightarrow B$ such that the composition of ψ with ϕ is an isogeny.

Exercise 14.7 (*). Construct an exact sequence $0 \longrightarrow U \longrightarrow W \longrightarrow V \longrightarrow 0$ of rational Hodge structures of weight 1 which is not split.

Exercise 14.8. Let h be real a quadratic form on a space $V_{\mathbb{R}}$ equipped with a Hodge structure $V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$. Prove that h is $U(1)$-invariant if and only if $h(x,y) = 0$ for any $x \in V^{p,q}, x \in V^{p',q'}$ unless $p = q'$ and $q = p'$.

Definition 14.3. Polarization on a rational Hodge structure of weight w is a $U(1)$-invariant non-degenerate 2-form $h \in V_{\mathbb{Q}}^* \otimes V_{\mathbb{Q}}^*$ (symmetric or antisymmetric depending on parity of w) which satisfies

$$-\sqrt{-1}^{\,p-q} h(x, \bar{x}) > 0 \qquad (*)$$

("Riemann-Hodge relations") for each non-zero $x \in V^{p,q}$. **Morphism of polarized Hodge structures** is a morphism of Hodge structures admitting polarization; no compatibility with the polarization is required.

Exercise 14.9. Prove that any exact sequence of polarizable rational Hodge structures is split.

Remark 14.2. The next exercise uses **Kodaira embedding theorem**, which says that a compact Kähler manifold is projective if and only if it admits a Kähler form with rational cohomology class. Feel free to use this theorem without a proof.

Definition 14.4. An abelian variety is a compact complex torus which is projective, that is, admits a holomorphic embedding to $\mathbb{C}P^n$.

Exercise 14.10. Let M be an n-dimensional abelian variety, and ω the Kähler class. Consider the form h on $H^1(M)$ taking η, η' to $\int_M \eta \wedge \eta' \wedge \omega^{n-1}$. Prove that h is a polarization.

Exercise 14.11 (*). Let M be a compact complex torus. Prove that M is an abelian variety if and only if the Hodge structure on its first cohomology admits a polarization.

Exercise 14.12 (*). Let $\phi: \ A \longrightarrow B$ be a surjective holomorphic map of abelian varieties. Prove that there exists a holomorphic map $\psi: \ B \longrightarrow A$ such that the composition of ψ and ϕ is an isogeny.

Hint. Use Exercise 14.6.

14.2 Hodge structures of K3 type

Exercise 14.13 (*). Let M be a projective complex surface[1] and $\omega \in H^{1,1}(M)$ a rational Kähler class. Consider $H^2(M)$ as a vector space equipped with the scalar product h, and let $H^2_{pr}(M)$ the space of all cohomology classes orthogonal to ω. Prove that h defines a polarization on the space $H^2_{pr}(M)$ equipped with a natural rational Hodge structure (the Hodge decomposition on $H^2_{pr}(M)$ comes from the Hodge theory).

Remark 14.3. For this exercise, it seems that a bit of Hodge theory is necessary.

Definition 14.5. **A Hodge structure of K3-type** is a Hodge structure $V_{\mathbb{C}} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$ of weight 2, with $\dim_{\mathbb{C}} V^{2,0} = 1$.

Definition 14.6. Consider a rational weight 2 Hodge structure on $V_{\mathbb{R}}$. A non-degenerate, rational, $U(1)$-invariant form on $V_{\mathbb{R}}$ is called **pseudopolarization** if $h(\xi, \bar{\xi}) > 0$ for any $\xi \in V^{2,0}$.

Exercise 14.14. Let M be a Kähler complex surface. Prove that the Poincaré pairing defines a pseudopolarization on $H^2(M)$.

Exercise 14.15. Let $V_{\mathbb{R}}$ be a vector space equipped with a non-degenerate bilinear symmetric form h.

a. Prove that a pseudo-polarized Hodge structure of K3 type is uniquely determined by h and the line $l = V^{2,0} \subset V_{\mathbb{C}}$.

b. Assume that $\xi \in V_{\mathbb{C}}$ is a vector which satisfies $h(\xi, \xi) = 0$ and $h(\xi, \bar{\xi}) > 0$. Prove that there $(V_{\mathbb{Q}}, h)$ admits a unique pseudo-polarized Hodge structure of K3 type such that $V^{2,0} = \mathbb{C} \cdot l$.

Definition 14.7. Let $V_{\mathbb{R}}$ be a vector space equipped with a non-degenerate bilinear symmetric form h. **The period space** $\mathbb{P}\mathsf{er}$ of $V_{\mathbb{R}}$ is the subset of $\mathbb{P}V_{\mathbb{C}}$ spanned by all vectors $\{p \in V_{\mathbb{C}} \ \mid \ h(p,p) = 0, h(p, \bar{p}) > 0\}$.

[1]A complex surface is a compact complex manifold of complex dimension 2.

Remark 14.4. The previous exercise establishes a bijective correspondence between $\mathbb{P}\mathsf{er}$ and the set of all pseudo-polarized Hodge structures of K3 type on $(V_{\mathbb{R}}, h)$.

Exercise 14.16. Let $V_{\mathbb{R}}$ be a vector space equipped with a non-degenerate bilinear symmetric form h, and $W_{\mathbb{R}} \subset V_{\mathbb{R}}$ a 2-dimensional real subspace such that $h\Big|_{W_{\mathbb{R}}}$ is non-degenerate. We extend h to a complex-linear pairing on $V_{\mathbb{C}}$.

a. Prove that $W_{\mathbb{C}} := W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ contains precisely 2 1-dimensional vector spaces W_1, W_2 such that $h\Big|_{W_i} = 0$.

b. Prove that the map $v \longrightarrow \operatorname{Re}(v)$ defines an isomorphism $W_i \longrightarrow W_{\mathbb{R}}$.

c. Prove that these projections define different orientations on $W_{\mathbb{R}}$.

d. (!) Construct an $O(V,h)$-invariant diffeomorphism between $\mathbb{P}\mathsf{er}$ and the Grassmannian $\operatorname{Gr}_{++}(V)$ of oriented 2-dimensional subspaces $W_{\mathbb{R}} \subset V_{\mathbb{R}}$ such that $h\Big|_{W_{\mathbb{R}}}$ is positive definite.

Definition 14.8. A Hodge structure is called **irreducible** if it does not have any Hodge substructures.

Exercise 14.17. Let V be an irreducible rational Hodge structure. Prove that the algebra $\mathcal{M}or(V,V)$ of morphisms from V to itself is a finite-dimensional division algebra over $\mathbb{Q}$.

Exercise 14.18. Let V be a rational, polarized Hodge structure of K3 type. Prove that $V = V_{tr} \oplus V_H$, where V_{tr} is an irreducible Hodge structure of K3 type, and $V_H \subset V^{1,1}$ a rational Hodge structure of Hodge type (1,1).

Definition 14.9. The Hodge substructure $V_{tr} \subset V$ is called **the transcendental Hodge lattice** and V_H **the lattice of Hodge cycles**.

Exercise 14.19. Let V be an irreducible Hodge structure of K3 type, and $\phi: V \longrightarrow V$ a Hodge morphism to itself which acts trivially on $V^{2,0}$. Prove that $\phi = \mathsf{Id}$.

Exercise 14.20. Let V be an irreducible Hodge structure of K3 type, and $E = \mathcal{M}or(V,V)$ its algebra of Hodge endomorphisms.

a. Prove that the restriction $E \longrightarrow \operatorname{Hom}(V^{2,0}, V^{2,0}) = \mathbb{C}$ defines an injective homomorphism from K to $\mathbb{C}$.

b. (!) Prove that E is a number field, that is, a finite field extension of $\mathbb{Q}$.

Hint. Use the previous exercise.