Hyperkähler manifolds (2023), the final exam

Rules: Every student gets 15 exercises (randomly chosen from this problem set), the final grade is determined by the score. Exercises are worth 10 points, unless indicated otherwise. The mark is C for score 40-59, B for 60-79, A for 80-99, A+ for the higher score. Please write down the solutions and bring them in person for discussion, no later than September 2023.

1 Holonomy groups, connections, Hodge theory

Exercise 1.1. Let M be a hyperkähler manifold with global holonomy $\operatorname{Sp}(n_1)\times\operatorname{Sp}(n_2)\times\ldots\times\operatorname{Sp}(n_k)$, with $2\sum_{i=1}^k n_i = \dim_{\mathbb{C}} M$. Prove that M is simply connected.

Exercise 1.2 (20 points). Find an example of a compact n-manifold M not admitting a Riemannian metric of signature $(2, n-2)$.

Exercise 1.3. Let η be a parallel form on a Riemannian manifold. Prove that η is harmonic.

Exercise 1.4 (20 points). Let η be a parallel 1-form on a Riemannian manifold, and ω a harmonic form. Prove that $\eta\wedge\omega$ is harmonic.

Exercise 1.5. Let (M, I) be an almost complex manifold, and ω a symplectic (1,1)-form such that $g := \omega(\cdot, I\cdot)$ is a positive definite scalar product on TM. Prove that ω is harmonic with respect to g.

Exercise 1.6. Construct a compact flat Kähler manifold with non-trivial canonical bundle.

Exercise 1.7 (30 points). Let M be a compact flat Kähler manifold. Prove that a finite covering of M is biholomorphic to a torus.

Exercise 1.8. Let G be a finite group which acts on a compact torus T by holomorphic diffeomorphisms. Assume that G preserves a holomorphic symplectic form. Prove that T admits a G-invariant hyperkähler metric.

Exercise 1.9 (30 points). Let G be an infinite group effectively acting by holomorphic automorphisms on a simply connected compact holomorphically symplectic Kähler manifold M. Prove that M does not admit a G-invariant Kähler metric.

2 Hypercomplex manifolds

Definition 2.1. An almost hypercomplex structure on a manifold M is a triple of almost complex structures (I, J, K) satisfying the quaternionic relations. It is called **hypercomplex** if I, J, K are integrable. **An almost hypercomplex Hermitian structure** on M is an almost complex structure (I, J, K) and a Riemannian metric h which is invariant under the action of I, J, K. It is called **hyperkähler** if I, J, K are complex, and h is Kähler with respect to I, J, K.

Exercise 2.1. Let M be a compact hypercomplex manifold, $\dim_{\mathbb{R}} M = 4n$ and η a $SU(2)$-invariant 2-form. Suppose that η has maximal rank in some point $m\in M$. Prove that $\int_M \eta^{2n}\neq 0$.

Exercise 2.2. Let be an almost hypercomplex manifold, $\dim_{\mathbb{R}} M = 4n$ and V the bundle of $SU(2)$-invariant 2-forms. Prove that $\operatorname{rk} V = \operatorname{rk}\operatorname{Sym}^2_{\mathbb{C}}\Lambda^{1,0}(M, I)$.

Exercise 2.3 (20 points). Let (M, I, K, K) be a compact hypercomplex manifold, and $\eta\in\Lambda^{2,0}(M, I)$ a (2,0)-form which satisfies $\partial\eta = 0$, $J\eta = \bar\eta$. Consider the standard action of $SU(2) = U(\mathbb{H}, 1)$ on $\Lambda^*(M)$. Prove that $d\eta$ belongs to a 2-dimensional irreducible $SU(2)$-representation.

Exercise 2.4 (20 points). Let (M, I, J, K) be a compact hypercomplex manifold, $\dim_{\mathbb H} M > 1$ and $V \subset \Lambda^{2,0}(M, I)$ a space of 2-forms which satisfy $\partial\eta = 0$, $J\eta = \bar\eta$. Prove that V is finite-dimensional, or find a counterexample.

Exercise 2.5 (30 points).
Let M be a compact hypercomplex manifold, $\dim_{\mathbb R} M = 4n$ and V the space of closed $SU(2)$-invariant 2-forms. Prove that V is finite-dimensional.

Exercise 2.6 (20 points). Let $\omega_1, \omega_2, \omega_3$ be a triple of 2-forms on a 4-manifold M such that any non-zero linear combination of ω_i is non-degenerate. Prove that there exists an almost hypercomplex Hermitian structure with fundamental forms $\omega_I, \omega_J, \omega_K$ such that the 3-dimensional sub-bundles of $\Lambda^2(M)$ spanned by $\omega_I, \omega_J, \omega_K$ and $\omega_1, \omega_2, \omega_3$ coincide.

Exercise 2.7 (20 points). Let (M, I, J, K) be a hypercomplex manifold, $d_I := IdI^{-1}, d_J := JdJ^{-1}, d_K := KdK^{-1}$, and $D := dd_Id_Jd_K : \Lambda^*(M) \longrightarrow \Lambda^{*+4}(M)$. Prove that D is independent from the choice of a basis I, J, K in quaternions, and its image is $SU(2)$-invariant.

Exercise 2.8 (20 points). Find an example of a compact hypercomplex manifold (M, I, J, K) such that (M, L) is biholomorphic to (M, I) for any induced complex structure L.

Exercise 2.9 (20 points). Let (M, I, J, K) be a compact hypercomplex manifold of real dimension 4, and ∇ **the Obata connection**, that is, the torsion-free connection on TM preserving I, J, K. Denote by η the curvature of the Obata connection on the canonical bundle $\Lambda^{2,0}(M, I)$; we consider η as a complex-valued 2-form on M. Prove that $\eta = 0$.

3 Hyperkähler manifolds

Exercise 3.1. Let (M, I, J, K) be an almost hypercomplex Hermitian manifold, and $\omega_I, \omega_J, \omega_K$ its fundamental forms. Suppose that these forms are closed. Prove that (M, I, J, K) is hyperkähler.

Exercise 3.2 (20 points). Let (M, I, J, K) be a hypercomplex Hermitian manifold, and $\omega_I, \omega_J, \omega_K$ its fundamental forms. Suppose that ω_I is closed. Prove that ω_J, ω_K are closed, or find a counterexample.

Exercise 3.3. Let (M, I, J, K) be an almost hypercomplex Hermitian manifold, and $\omega_I, \omega_J, \omega_K$ its fundamental forms. Suppose that ω_I, ω_J are closed. Prove that ω_K is closed, or find a counterexample.

a. (20 points) Find a solution when $\dim_{\mathbb C} M = 2$.

b. (30 points) Find a solution for any dimension.

Exercise 3.4 (20 points). Let $Z \subset (M, \Omega)$ be a submanifold in a holomorphically symplectic manifold. Assume that Z is Lagrangian with respect to the symplectic forms $Re\,\Omega$ and $Im\,\Omega$. Prove that Z is complex analytic.

Definition 3.1. A holomorphic Lagrangian fibration on a holomorphically symplectic manifold (M, Ω) is a holomorphic submersion $\pi : M \longrightarrow X$ such that the fibers of π are holomorphic Lagrangian with respect to Ω.

Exercise 3.5 (20 points). Let $\pi : M \longrightarrow X$ be a holomorphic Lagrangian fibration, and $\sigma : X \longrightarrow M$ a smooth section. Prove that $\sigma^*(\Omega)$ has Hodge type $(2, 0) + (1, 1)$.

Exercise 3.6 (20 points). Let $L \subset M$ be a complex Lagrangian submanifold in a compact hyperkähler manifold. Prove that L is projective. Use the Kodaira embedding theorem.

Exercise 3.7 (20 points). Let M be a compact hyperkähler manifold of maximal holonomy, and $M \longrightarrow B$ a surjective holomorphic map to a Kähler manifold B, $\dim B < \dim M$. Prove that B is projective. Use the Kodaira embedding theorem.

Exercise 3.8 (30 points). Let G be a finite group freely acting on a hyperkähler manifold M by holomorphic automorphisms. Prove that M/G is hyperkähler or projective. Use Kodaira embedding theorem.

Exercise 3.9 (30 points). Let M be a hyperkähler manifold, $\dim_{\mathbb{C}} M = 2$. Prove that the set of smooth complex curves of genus 0 on M is countable.

4 Trianalytic subvarieties

Exercise 4.1. Find a compact hypercomplex manifold (M, I, J, K) such that for any induced complex structure $L = aI + bJ + cK$, the manifold (M, L) has non-trivial divisors.

Exercise 4.2. Let (M, I, J, K) be a hyperkähler manifold, and ϕ - a holomorphic automorphism of (M, I) which acts trivially on $H^2(M)$. Prove that its graph is trianalytic.

Exercise 4.3 (20 points). Let (M, I, J, K) be a hyperkähler manifold, and $X \subset (M, I)$ a complex subvariety of dimension 2. Prove that $\int_X \omega_I^2 \geqslant \frac{1}{2} \int_X (\Omega \wedge \bar{\Omega})$. Prove that equality is realized if and only if X trianalytic.

Exercise 4.4 (20 points). Let $X \subset T^{4n}$ be a trianalytic subvariety in a compact torus. Prove that X is a union of subtori.

Exercise 4.5 (20 points). Let X be a hyperkähler manifold (not necessarily compact) with $b_2(X) = 1$. Prove that there exists an induced complex structure L on X such that (X, L) contains no compact complex subvarieties.

Exercise 4.6. Let X be a trianalytic submanifold in a product of two hyperkähler manifolds X_1 and X_2. Assume that X is a hyperkähler manifold of maximal holonomy. Prove that X belongs to a fiber of the projection $X_1 \times X_2 \longrightarrow X_1$ or to a fiber of the projection $X_1 \times X_2 \longrightarrow X_2$.

Exercise 4.7. Let (X, I) be a complex manifold; then T^*X is equipped with the standard holomorphic symplectic form Ω. Assume that T^*X admits a hyperkähler structure (I, J, K, g) such that $\Omega = \omega_J + \sqrt{-1}\omega_K$. Prove that T^*X does not contain compact trianalytic subvarieties.

Exercise 4.8. Consider a complete hyperkähler manifold (M, I, J, K, g), $\dim M = 4n$, equipped with an effective action of a compact n-dimensional torus T preserving the hyperkähler structure. Assume that all elements in the Lie algebra of T act on M by vector fields which are Hamiltonian with respect to $\omega_I, \omega_J, \omega_K$. Then (M, I, J, K, g) is called **a hypertoric manifold**.

a. Prove that a hypertoric manifold is never compact.

b. Construct an example of a hypertoric manifold.

Exercise 4.9 (30 points). Let (M, I, J, K, g) be a simply connected hyperkähler manifold equipped with a non-trivial action $\rho(t)$ of S^1 preserving g and I. Assume that $\rho(t)^*\Omega_I = e^{\sqrt{-1}\pi t}\Omega_I$, where $\Omega_I = \omega_J + \sqrt{-1}\omega_K$ is the holomorphic symplectic form on (M, I). Prove that the forms ω_J and ω_K are exact.

5 Fujiki formula, BBF form and automorphisms

Definition 5.1. An algebraic function $A:\ \mathbb{R}^n \longrightarrow \mathbb{R}$ is a continuous function which is given as one of the branches of a multi-valued function μ with the graph $\Gamma_\mu \subset \mathbb{R}^n \times \mathbb{R}$ which is an irreducible algebraic variety.

Exercise 5.1 (30 points). Let $A:\ \mathbb{R}^n \longrightarrow \mathbb{R}$ be an algebraic function, and $S \subset \operatorname{Gr}(k,n)$ an open subset in the Grassmannian, with $k > 1$. Assume that A is polynomial on each $l \in S$. Prove that A is polynomial.

Exercise 5.2 (20 points). Let M be a product of two compact hyperkähler manifolds. Prove that M cannot be homeomorphic to a hyperkähler manifold of maximal holonomy.

Exercise 5.3 (20 points). Let ϕ be a complex authomorphism of a compact hyperkähler manifold (M,I) of maximal holonomy. Assume that the action of ϕ on $H^{1,1}_I(M)$ has a fixed point v such that $q(v,v) > 0$. Prove that v or $-v$ is a Kähler class on (M,I).

Exercise 5.4. Let ϕ be a complex authomorphism of a compact hyperkähler manifold (M,I) of maximal holonomy.

a. (20 points) Assume that ϕ preserves a Kähler class $u \in H^{1,1}_I(M)$. Prove that ϕ has finite order.

b. (30 points) Let $\lambda := \lim_i \left\|\phi^i\right\|^{1/i}$, where $\|\phi^i\|$ denotes the operator norm of $\phi\Big|_{H^{1,1}_I(M)}$. Assume that $\lambda > 1$. Prove that $\lim\limits_{i\to\infty} \frac{\phi^i(u)}{\lambda^i}$ converges to a point v on the boundary of the Kähler cone. Prove that v is determined uniquely up to a scalar multiplier by the automorphism ϕ. Prove that $q(v,v) = 0$.

Exercise 5.5. Let (M,I) be a compact hyperkähler manifold of maximal holonomy, and Ω its holomorphic symplectic form.

a. Let $Z \subset (M,I)$ be a complex submanifold. Prove that $\int_Z \Omega \wedge \bar\Omega \wedge \omega^{\dim Z - 2} \geqslant 0$, with equality $\Leftrightarrow \Omega\big|_Z = 0$

b. Denote by $[\Omega]$ the cohomology class of the holomorphically symplectic form on (M,I). Prove that a complex submanifold $Z \subset (M,I)$, $\dim Z = 1/2 \dim M$ is holomorphic Lagrangian if and only if $[Z] \wedge [\Omega] = 0$, where $[Z] \in H^*(M)$ denotes the fundamental class of Z.

c. (30 points) Consider a holomorphic embedding $(M,I) \hookrightarrow \mathbb{C}P^n$. Let $H \subset \mathbb{C}P^n$ be a complex submanifold which has transversal intersection with (M,I). Prove that $H \cap (M,I)$ is never holomorphic Lagrangian.

Exercise 5.6 (30 points). Let ϕ be a complex authomorphism of a compact hyperkähler manifold (M,I) of maximal holonomy. Assume that (M,I) is projective. Prove that there exists $d > 0$ such that ϕ^d preserves the holomorphic symplectic form.

Exercise 5.7 (20 points). Let ϕ be a complex authomorphism of a compact hyperkähler manifold (M,I) of maximal holonomy. Assume that ϕ preserves a complex curve $S \subset (M,I)$. Prove that the eigenvalues α_i of the restriction $\phi\Big|_{H^{1,1}(M,I)}$ satisfy $|\alpha_i| = 1$.

Exercise 5.8 (30 points). Let ϕ be a complex authomorphism of a compact hyperkähler manifold (M,I) of maximal holonomy. Assume that (M,I) is projective and $\operatorname{rk} H^{1,1}(M,I) \cap H^2(M,\mathbb{Z}) = 1$. Prove that $\phi\Big|_{H^2(M)}$ has finite order.