Hyperkahler manifolds,

lecture 3: Frobenius theorem

IMPA, sala 236

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Distributions

DEFINITION: Distribution on a manifold is a sub-bundle $B \subset TM$

REMARK: Let $\Pi: TM \longrightarrow TM/B$ be the projection, and $x, y \in B$ some vector fields. Then $[fx, y] = f[x, y] - D_y(f)x$. This implies that **$\Pi([x, y])$ is $C^\infty(M)$-linear as a function of x and y.**

DEFINITION: The map $[B, B] \longrightarrow TM/B$ we have constructed is called **Frobenius bracket** (or **Frobenius form**); it is a skew-symmetric $C^\infty(M)$-linear form on B with values in TM/B.

DEFINITION: A distribution is called **holonomic**, or **involutive**, if its Frobenius form vanishes.

Smooth submersions

DEFINITION: Let $\pi : M \longrightarrow M'$ be a smooth map of manifolds. This map is called **submersion** if at each point of M the differential $D\pi$ is surjective, and **immersion** if it is injective.

CLAIM: Let $\pi : M \longrightarrow M'$ be a submersion. Then each $m \in M$ has a neighbourhood $U \cong V \times W$, where V, W are smooth and $\pi|_U$ **is a projection of $V \times W = U \subset M$ to $W \subset M'$ along V.**

Proof: Follows from the inverse function theorem. ■

THEOREM: ("Ehresmann's fibration theorem")
Let $\pi : M \longrightarrow M'$ be a smooth submersion of compact manifolds. **Prove that π is a locally trivial fibration.**

Proof: Next slide.

DEFINITION: Vertical tangent space $T_\pi M \subset TM$ of a submersion $\pi : M \longrightarrow M'$ is the kernel of $D\pi$.

Ehresmann connections

DEFINITION: Let $\pi: M \longrightarrow Z$ be a smooth submersion, with $T_\pi M$ **the bundle of vertical tangent vectors** (vectors tangent to the fibers of π). An **Ehresmann connection** on π is a sub-bundle $T_{\text{hor}}M \subset TM$ such that $TM = T_{\text{hor}}M \oplus T_\pi M$. The **parallel transport** along the path $\gamma: [0,a] \longrightarrow Z$ associated with the Ehresmann connection is a diffeomorphism

$$V_t: \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(t))$$

smoothly depending on $t \in [0,a]$ and satisfying $\frac{dV_t}{dt} \in T_{\text{hor}}M$.

CLAIM: Let $\pi: M \longrightarrow Z$ be a smooth fibration with compact fibers. Then **the parallel transport, associated with the Ehresmann connection, always exists.**

Proof: Follows from existence and uniqueness of solutions of ODEs. ■

Foliations

Frobenius Theorem: Let $B \subset TM$ be a sub-bundle. Then B is involutive if and only if each point $x \in M$ has a neighbourhood $U \ni x$ and **a smooth submersion $U \stackrel{\pi}{\longrightarrow} V$ such that B is its vertical tangent space: $B = T_\pi M$.**

REMARK: The implication **"$B = T_\pi M$" $\Rightarrow$ "Frobenius form vanishes"** is clear because of local coordinate form of the submersions.

DEFINITION: The fibers of π are called **leaves**, or **integral submanifolds** of the distribution B. Globally on M, **a leaf of B** is a maximal connected manifold $Z \hookrightarrow M$ which is immersed to M and tangent to B at each point. A distribution for which Frobenius theorem holds is called **integrable**. If B is integrable, the set of its leaves is called **a foliation**. The leaves are manifolds which are immersed to M, but not necessarily closed.

REMARK: To prove the Frobenius theorem for $B \subset TM$, **it suffices to show that each point is contained in an integral submanifold.** In this case, the smooth submersion $U \stackrel{\pi}{\longrightarrow} V$ is the projection to the leaf space of B.

Frobenius theorem (1)

Proof of the Frobenius theorem. Step 1: Suppose that G is a Lie group acting on a manifold M. Assume that the vector fields from the Lie algebra of G generate a sub-bundle $B \subset TM$. **Then B is integrable, that is, Frobenius theorem holds of $B \subset TM$.** Indeed, the orbits of the G-action are tangent to $B \subset TM$.

Step 2: Let u, v be commuting vector fields on a manifold M, and e^{tu}, e^{tv} be corresponding diffeomorphism flows. **Then e^{tu}, e^{tv} commute.** This easily follows by taking a coordinate system such that u is the coordinate vector field.

Step 3: The commutator of vector fields in B belongs to B, however, this does not immediately produce any finite-dimensional Lie algebra: it is not obvious that any subalgebra generated by such vector fields is finite-dimensional. To produce a Lie group with orbits tangent to B, **we need to find a collection $\xi_1, ..., \xi_k \in B$ of vector fields generating B and make sure that the $\xi_1, ..., \xi_k$ generate a finite-dimensional Lie algebra.**

Frobenius theorem (2)

Step 4: The statement of Frobenius Theorem is local, hence we may replace M be a small neighbourhood of a given point. **We are going to show that B locally has a basis of commuting vector fields.** By Step 2, these vector fields can be locally integrated to a commutative group action, and Frobenius Theorem follows from Step 1.

Step 5: Let $\sigma : M \longrightarrow M_1$ be a smooth submersion, $d\sigma : T_x M \longrightarrow T_{\sigma(x)} M_1$ its differential, and $v \in TM$ a vector field which satisfies

$$d\sigma(v)|_x = d\sigma(v)|_y \quad (*)$$

for any $x, y \in \sigma^{-1}(z)$ and any $z \in M_1$. In this case, the vector field $d\sigma(v)$ is well-defined on M_1. **Given two vector fields u and v which satisfy (*), we can easily check that the commutator $[u, v]$ also satisfies (*),** and, moreover, $d\sigma([u, v]) = [d\sigma(u), d\sigma(v)]$.

Frobenius theorem (3)

Step 6: Now we can finish the proof of Frobenius theorem. We need to produce, locally in M, a basis of commuting vector fields $\xi_i \in B$. **We start with producing (locally in M) an auxiliary submersion σ, with the fibers which are complementary to B.** To define such a submersion, we put coordinates locally on M, identifying M with an open subset in $\mathbb{R}^n$, and take a linear map $\sigma: M \longrightarrow M_1 = \mathbb{R}^{\dim B}$ such that $d\sigma: B|_x \longrightarrow T_{\sigma(x)}M_1$ is an isomorphism at some $x \in M$.

Step 7: Then $d\sigma: B|_x \tilde{\longrightarrow} T_{\sigma(x)}M_1$ is an isomorphism in a neighbourhood of x; replacing M by a smaller open set, we may assume that $d\sigma: B|_x \tilde{\longrightarrow} T_{\sigma(x)}M_1$ is an isomorphism everywhere on M. Let $\zeta_1, ..., \zeta_k$ be the coordinate vector fields on M_1.

Since $d\sigma: B|_x \longrightarrow T_{\sigma(x)}M_1$ is an isomorphism, there exist unique vector fields $\xi_1, ..., \xi_k \in B \subset TM$ such that $d\sigma(\xi_i) = \zeta_i$. By Step 5, $d\sigma([\xi_i, \xi_j]) = [\zeta_i, \zeta_j] = 0$. Since B is involutive, the commutator $[\xi_i, \xi_j]$ is a section of B. Now, the map $d\sigma: B|_x \longrightarrow T_{\sigma(x)}M_1$ is an isomorphism, and therefore the vanishing of $d\sigma([\xi_1, \xi_j])$ implies $[\xi_1, \xi_j] = 0$. **We have constructed a basis of commuting vector fields in B and finished the proof of Frobenius theorem.** ■