Hyperkähler manifolds,

lecture 16: Complex subvarieties

IMPA, sala 236

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Complex subvarieties

The aim of today's lecture is

THEOREM: Let (M,I) be a complex manifold, $Z \subset (M,I)$ a real analytic subvariety, and $Z_0 \subset Z$ the set of smooth points of Z. **Suppose that $I(TZ_0) = TZ_0$. Then Z is complex analytic.**

Plan: Define real analytic varieties and manifolds, prove Newlander-Nirenberg for real analytic complex structures, use it to prove this result.

Real structures on complex manifolds

DEFINITION: A smooth map $\Psi: M \longrightarrow N$ on an almost complex manifold (M, I) is called **antiholomorphic** if $d\Psi(I) = -I$. A function f is called **antiholomorphic** if $\overline{f}$ is holomorphic.

EXERCISE: Prove that **an antiholomorphic function on M defines an antiholomorphic map from M to $\mathbb{C}$.**

EXERCISE: Prove that a map $\Psi: M \longrightarrow N$ of almost complex manifolds is antiholomorphic **if and only if $\Psi^*(\Lambda^{0,1}(N)) \subset \Lambda^{1,0}(M)$.**

EXERCISE: Let ι be a smooth map from a complex manifold M to itself. Prove that ι **is antiholomorphic if and only if $\iota^*(f)$ is antiholomorphic for any holomorphic function f on $U \subset M$.**

DEFINITION: A real structure on a complex manifold M is an antiholomorphic involution $\tau: M \longrightarrow M$.

EXAMPLE: Complex conjugation defines a real structure on $\mathbb{C}^n$.

Fixed points of real structures on manifolds

PROPOSITION: Let M be a complex manifold and $\iota: \ M \longrightarrow M$ a real structure. Denote by M^ι the fixed point set of ι. Then, **for each $x \in M^\iota$ there exists a ι-invariant coordinate neighbourhood with holomorphic coordinates $z_1, ..., z_n$, such that $\iota^*(z_i) = \overline{z}_i$.**

Proof. Step 1: For each basis of 1-forms $\nu_1, ..., \nu_n \in \Lambda^{1,0}_x(M)$, there exists a set of holomorphic coordinate functions $u_1, ..., u_n$ such that $du_i|_x = \nu_i$. To obtain such a coordinate system, **we chose any coordinate system $v_1, ..., v_n$ and apply a linear transform mapping $dv_i|_x$ to ν_i.**

Step 2: The differential $d\iota$ acts on T_xM as a real structure. Using the structure theorem about real structures, we obtain that any real basis $\zeta_1, .., \zeta_n$ of $T^*_x M^\iota$ is a complex basis in the complex vector space $T^*_x M$. Then $\nu_i := \zeta_i + \sqrt{-1}\, I(\zeta_i)\}$ is a basis in $\Lambda^{1,0}_x(M)$. Choose the coordinate system $u_1, ..., u_n$ such that $du_i|_x = \nu_i$ (Step 1). **Replacing u_i by $z_i := u_i + \iota^*(\overline{u}_i)$, we obtain a holomorphic coordinate system z_i on M** (compare with Theorem 1 in Lecture 4) **which satisfies $\iota^*(z_i) = \overline{z}_i$.** ■

DEFINITION: Let $\{U_i\}$ be an complex atlas on M. Assume that any U_i intersecting M^ι satisfies the conclusion of this proposition. Then $\{U_i\}$ is called **compatible with the real structure**.

Real analytic manifolds and real structures

PROPOSITION: Let $M^\iota \subset M$ be a fixed point set of an antiholomorphic involution ι on a complex manifold M, $\{U_i\}$ a complex analytic atlas, and $\Psi_{ij}: U_{ij} \longrightarrow U_{ij}$ the gluing functions. Assume that the atlas U_i is compatible with the real structure, in the sense of the previous proposition. **Then all Ψ_{ij} are real analytic on M^ι, and define a real analytic atlas on the manifold M^ι.**

Proof: All gluing functions from one coordinate system compatible with the real structure to another **commute with ι, acting on coordinate functions as the complex conjugation.** This gives $\Psi_{ij}(\overline{z}_i) = \overline{\Psi_{ij}(z_i)}$. Therefore, Ψ_{ij} preserve M^ι, and are expressed by real-valued functions on M^ι. ■

Real analytic manifolds and real structures 2

PROPOSITION: Any real analytic manifold can be obtained from this construction.

Proof. Step 1: Let $\{U_i\}$ be a locally finite atlas of a real analytic manifold M, and $\Psi_{ij}: U_{ij} \longrightarrow U_{ij}$ the gluing maps. We realize U_i as an open ball with compact closure in $\operatorname{Re}(\mathbb{C}^n) = \mathbb{R}^n$. By local finiteness, there are only finitely many such Ψ_{ij} for any given U_i. Denote by B_ε an open ball of radius ε in the n-dimensional real space $\operatorname{im}(\mathbb{C}^n)$.

Step 2: Let $\varepsilon > 0$ be a sufficiently small real number such that all Ψ_{ij} can be extended to gluing functions $\tilde{\Psi}_{ij}$ on the open sets $\tilde{U}_i := U_i \times B_\varepsilon \subset \mathbb{C}^n$. **Then $(\tilde{U}_i, \Psi_{ij})$ is an atlas for a complex manifold $M_{\mathbb{C}}$.** Since all Ψ_{ij} are real, they are preserved by the natural involution acting on B_ε as -1 and on U_i as identity. This involution defines a real structure on $M_{\mathbb{C}}$. Clearly, M is the set of its fixed points. ■

Complexification

DEFINITION: Let $M_{\mathbb{R}}$ be a real analytic manifold, and $M_{\mathbb{C}}$ a complex analytic manifold equipped with an antiholomorphic involution, such that $M_{\mathbb{R}}$ is the set of its fixed points. Then $M_{\mathbb{C}}$ is called **complexification** of $M_{\mathbb{R}}$.

DEFINITION: A tensor on a real analytic manifold is called **real analytic** if it is expressed locally by a sum of coordinate monomials with real analytic coefficients.

CLAIM: Let $M_{\mathbb{R}}$ be a real analytic manifold, $(M_{\mathbb{C}}, \iota)$ its complexification, and Φ a tensor on $M_{\mathbb{R}}$. **Then Φ is real analytic if and only if Φ can be extended to a holomorpic tensor $\Phi_{\mathbb{C}}$ in some neighbourhood of $M_{\mathbb{R}}$ inside $M_{\mathbb{C}}$.** Moreover, **Φ is real on $M_{\mathbb{R}}$ if $\iota^*\Phi_{\mathbb{C}} = \overline{\Phi}_{\mathbb{C}}$.**

Proof: The "if" part is clear, because every complex analytic tensor on $M_{\mathbb{C}}$ is by definition real analytic on $M_{\mathbb{R}}$.

Conversely, suppose that Φ is expressed in coordinates by a sum of tensorial monomials with real analytic coefficients f_i. Let $\{U_i\}$ be a cover of M, and $\tilde{U}_i := U_i \times B_\varepsilon$ the corresponding cover of a neighbourhood of $M_{\mathbb{R}}$ in $M_{\mathbb{C}}$ constructed above. Chosing ε sufficiently small, we can assume that the Taylor series giving coefficients of Φ converges on each $\tilde{U}_i$. **We define $\Phi_{\mathbb{C}}$ as the sum of these series.** ∎

Germ of a complex manifold

DEFINITION: Let $K\subset M$ be a closed subset of a complex manifold, homeomorphic to $K_1\subset M_1$, where M_1 is also a complex manifold. Fixing the homeomorphism $K\cong K_1$, we may identify these sets and consider K as a subset M_1. We say that M and M_1 **have the same germ in K** if there exist biholomorphic open subsets $U_1\subset M_1$ and $U\subset M$ containing K, with the biholomorphism $\varphi: U\longrightarrow U_1$ identity on K.

DEFINITION: Germ of a manifold M in $K\subset M$ is an equivalence class of open subsets $U\subset M$ containing K, with this equivalence relation.

DEFINITION: Consider category $\mathcal{C}_\iota$, with objects complex manifolds (M,ι) equipped with a real structure, and morphisms holomorphic maps commuting with ι.

THEOREM: (Grauert) Category of real analytic manifolds is equivalent to the category of germs of $M\in\mathcal{C}_\iota$ in $M^\iota\subset M$.

EXERCISE: Prove this theorem.

Hans Grauert

Hans Grauert in Bonn, 2000

(8.02.1930 - 4.09.2011)

Extension of tensors to a complexification

Lemma 1: Let X be an open ball in $\mathbb{C}^n$ equipped with the standard anticomplex involution, $X_{\mathbb{R}} = X \cap \mathbb{R}^n$ its fixed point set, and α a holomorphic tensor on X vanishing in $X_{\mathbb{R}}$. **Then $\alpha = 0$.**

Proof: Any holomorphic function which vanishes on $\mathbb{R}^n$ has all its derivatives vanishing. Therefore its Taylor serie vanish. Such a function vanishes on $\mathbb{C}^n$ by analytic continuation principle. This argument can be applied to all coefficients of α. ■

DEFINITION: An almost complex structure I on a real analytic manifold is **real analytic** if I is a real analytic tensor.

COROLLARY: Let (M,I) be a real analytic almost complex manifold, $M_{\mathbb{C}}$ its complexification, and $I_{\mathbb{C}}: TM_{\mathbb{C}} \longrightarrow TM_{\mathbb{C}}$ the holomorphic extension of I to $M_{\mathbb{C}}$. **Then $I_{\mathbb{C}}^2 = -\operatorname{Id}$.**

Proof: The tensor $I_{\mathbb{C}}^2 + \operatorname{Id}$ is holomorphic and vanishes on $M_{\mathbb{R}}$, hence the previous lemma can be applied. ■

Underlying real analytic manifold

REMARK: A complex analytic map $\Phi: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is real analytic as a map $\mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$. Indeed, the coefficients of Φ are real and imaginary parts of holomorphic functions, and real and imaginary parts of holomorphic functions can be expressed as Taylor series of the real variables.

DEFINITION: Let M be a complex manifold. The **underlying real analytic manifold** $M_{\mathbb{R}}$ is the same manifold, with the same gluing functions, considered as real analytic maps.

REMARK: The sheaf of real analytic functions on $M_{\mathbb{R}}$ can be defined as **the sheaf of converging power series generated by holomorphic and antiholomorphic functions.** Indeed, such functions are real analytic in any of the real analytic map; conversely, **any real analytic function on $M_{\mathbb{R}}$ is a converging power serie on $\operatorname{Re} z_i, \operatorname{Im} z_i$, where z_i are holomorphic coordinates on M.**

Complexification of the underlying real analytic manifold

DEFINITION: Let M be a complex manifold. The **complex conjugate manifold** is the same manifold with almost complex structure $-I$ and anti-holomorphic functions on M holomorphic on $\overline{M}$.

CLAIM: Let M be an integrable almost complex manifold. Denote by $M_{\mathbb{R}}$ its underlying real analytic manifold. **Then a complexification of $M_{\mathbb{R}}$ can be given as $M_{\mathbb{C}} := M \times \overline{M}$,** with the anticomplex involution $\tau(x, y) = (y, x)$.

Proof: Clearly, the fixed point set of τ is the diagonal, identified with $M_{\mathbb{R}} = M$ as usual. Both holomorphic and antiholomorphic functions on $M_{\mathbb{R}}$ are obtained as restrictions of holomorphic functions from $M_{\mathbb{C}}$, hence the sheaf of real analytic functions on $M_{\mathbb{R}}$ is a subsheaf of $\mathcal{O}_{M_{\mathbb{C}}}$ of holomorphic functions on $M_{\mathbb{C}}$ restricted to $M_{\mathbb{R}}$. ■

Holomorphic and antiholomorphic foliations

DEFINITION: Let $B \subset TM$ be a sub-bundle. The **foliation associated with B** is a family of submanifolds $X_t \subset U$, defined for each sufficiently small subset of M, called **the leaves of the foliation**, such that B is the bundle of vectors tangent to X_t. In this case, X_t are called **the leaves** of the foliation.

REMARK: Frobenius theorem says that B **is involutive if and only if it is tangent to a foliation.**

REMARK: Let (M, I) be a real analytic almost complex manifold, and $M_{\mathbb C}$ its complexification. Replacing $M_{\mathbb C}$ by a smaller neighbourhood of M, we may assume that the tensor I is extended to an endomorphism $I: TM_{\mathbb C} \longrightarrow TM_{\mathbb C}$, $I^2 = -\operatorname{Id}$. **Since $TM_{\mathbb C}$ is a complex vector bundle, I acts there with the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, giving a decomposition $TM_{\mathbb C} = T^{1,0}M_{\mathbb C} \oplus T^{0,1}M_{\mathbb C}$**

DEFINITION: **Holomorphic foliation** is a foliation tangent to $T^{1,0}M_{\mathbb C}$, **antiholomorphic foliation** is a foliation tangent to $T^{0,1}M_{\mathbb C}$.

Antiholomorphic foliation on $M_{\mathbb C} = M \times \overline{M}$.

REMARK: Let (M,I) be a integrable almost complex manifold, $M_{\mathbb C} = M \times \overline{M}$ its complexification, and $\pi, \overline{\pi}$ projections of $M_{\mathbb C}$ to M and $\overline{M}$. **Then the fibers of $\overline{\pi}$ is a holomorphic foliation, and the fibers of π is a holomorphic foliation.**

REMARK: Let $TM_{\mathbb C} = T' \oplus T''$ be a decomposition of $TM_{\mathbb C}$ onto part tangent to fibers of $\overline{\pi}$ and tangent to fibers of π. **On $M_{\mathbb R}$ the decomposition $TM_{\mathbb C} = T' \oplus T''$ coincides with the decomposition $TM \otimes \mathbb C = T^{1,0}M \oplus T^{0,1}M$.**

COROLLARY: Let (M,I) be a integrable almost complex manifold. **Then I is a real analytic almost complex structure.**

Proof: Extend I to an operator on $M_{\mathbb C}$ acting as $\sqrt{-1}$ on T' and $-\sqrt{-1}$ on T''. This operator is complex analytic because the decomposition $TM = T' \oplus T''$ is holomorphic. ■

Corollary 1: Let (M,I) be a real analytic almost complex manifold. Then holomorphic functions on $M_{\mathbb C}$ which are constant on the leaves of antiholomoirphic foliation **restrict to holomorphic functions on $(M,I) \subset M_{\mathbb C}$.**

Proof: Such functions are constant in the $(0,1)$-direction on $TM \otimes \mathbb C$. ■

Integrability of real analytic almost complex structure

THEOREM: (Newlander-Nirenberg for real analytic manifolds) Let (M, I) be a real analytic almost complex manifold, $\dim_{\mathbb{R}} M = 2$. **Then M is integrable.**

Proof. Step 1: Consider the complexification $M_{\mathbb{C}}$ of M, and let $TM_{\mathbb{C}} = T^{1,0}M_{\mathbb{C}} \oplus T^{0,1}M_{\mathbb{C}}$ be the decomposition defined above. By Frobenius theorem, there exists a foliation tangent to $T^{0,1}M_{\mathbb{C}}$ and one tangent to $T^{1,0}M_{\mathbb{C}}$. Since the leaves of these foliations are transversal, **locally $M_{\mathbb{C}}$ is a product of M' and M'' which are identified with the space of leaves of $T^{0,1}M_{\mathbb{C}}$ and $T^{1,0}M_{\mathbb{C}}$.**

Step 2: Locally, functions on M' can be lifted to $M' \times M'' = M_{\mathbb{C}}$, giving functions which are constant on the leaves of the foliation tangent to $T^{0,1}M_{\mathbb{C}}$. By Corollary 1, such functions are holomorphic on (M, I). Choose a collection of $n = \frac{1}{2}\dim_{\mathbb{R}} M$ holomorphic functions $f_1, ... f_n$ on $M_{\mathbb{C}}$ which are constant on the leaves of $T^{0,1}M_{\mathbb{C}}$ and have linearly independent differentials in $x \in M \subset M_{\mathbb{C}}$. By inverse function theorem, **$f_1, ..., f_n$ is a holomorphic coordinate system in a neigbourhood of $x \in (M, I)$,** and the transition functions between such coordinate systems are by construction holomorphic. ■

Complex subvarieties

CLAIM: Let $Z \subset M$ be a real analytic subvariety in a complex manifold (M, I), and $Z_{\mathbb{C}} \subset M_{\mathbb{C}} = (M, I) \times (M, -I)$ its complexification, defined locally in a neighbourhood of a given point $z \in Z$. Assume that $Z_{\mathbb{C}}$ is locally a product, $Z_{\mathbb{C}} = Z' \times Z''$, with $Z' \subset (M, I)$ and $Z'' \subset (M, -I)$. **Then Z is complex analytic.**

Proof: The projection $Z \longrightarrow Z'$ is locally bijective, and identifies Z with a complex variety. These local isomorphisms define complex analytic charts on Z, commuting with the projection $M_{\mathbb{C}} \longrightarrow (M, I)$, and hence holomorphic. ■

THEOREM: Let (M, I) be a complex manifold, $Z \subset (M, I)$ a real analytic subvariety, and $Z_0 \subset Z$ the set of smooth points of Z. **Suppose that $I(TZ_0) = TZ_0$. Then Z is complex analytic.**

Proof: By the previous claim, it would suffice to check that $Z_{\mathbb{C}}$ is locally a product, $Z_{\mathbb{C}} = Z' \times Z''$, with $Z' \subset (M, I)$ and $Z'' \subset (M, -I)$. A closure of a product is a product, and $(Z_0)_{\mathbb{C}} = Z_0' \times Z_0''$ because it is complex analytic. ■