Hyperkähler manifolds,

lecture 20: Yang-Mills connections

IMPA, sala 236

Misha Verbitsky, May 22, 2023, 13:30

http://verbit.ru/IMPA/HK-2023/

REMINDER: Holomorphic vector bundles

DEFINITION: A $\overline{\partial}$-operator on a smooth bundle is a map $V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V$, satisfying $\overline{\partial}(fb) = \overline{\partial}(f) \otimes b + f\overline{\partial}(b)$ for all $f \in C^\infty M, b \in V$.

REMARK: A $\overline{\partial}$-operator on B can be extended to

$$\overline{\partial} : \Lambda^{0,i}(M) \otimes V \longrightarrow \Lambda^{0,i+1}(M) \otimes V,$$

using $\overline{\partial}(\eta \otimes b) = \overline{\partial}(\eta) \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \overline{\partial}(b)$, where $b \in V$ and $\eta \in \Lambda^{0,i}(M)$.

DEFINITION: A holomorphic vector bundle on a complex manifold (M, I) is a vector bundle equipped with a $\overline{\partial}$-operator which satisfies $\overline{\partial}^2 = 0$. In this case, $\overline{\partial}$ is called **a holomorphic structure operator**.

EXERCISE: Consider the Dolbeault differential $\overline{\partial} : \Lambda^{p,0}(M) \longrightarrow \Lambda^{p,1}(M) = \Lambda^{p,0}(M) \otimes \Lambda^{0,1}(M)$. **Prove that it is a holomorphic structure operator on $\Lambda^{p,0}(M)$.**

DEFINITION: The corresponding holomorphic vector bundle $(\Lambda^{p,0}(M), \overline{\partial})$ is called **the bundle of holomorphic p-forms**, denoted by $\Omega^p(M)$.

REMINDER: Chern connection

DEFINITION: Let (B, ∇) be a smooth bundle with connection and a holomorphic structure $\overline{\partial}\, B \longrightarrow \Lambda^{0,1}(M) \otimes B$. Consider a Hodge decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$,

$$\nabla^{0,1} : V \longrightarrow \Lambda^{0,1}(M) \otimes V, \quad \nabla^{1,0} : V \longrightarrow \Lambda^{1,0}(M) \otimes V.$$

We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: **An Hermitian holomorphic vector bundle** is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure operator $\overline{\partial}$.

DEFINITION: **A Chern connection** on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

Kobayashi-Hitchin correspondence

DEFINITION: Let F be a coherent sheaf over an n-dimensional compact Kähler manifold M. Let

$$\operatorname{slope}(F) := \frac{1}{\operatorname{rank}(F)} \int_M \frac{c_1(F) \wedge \omega^{n-1}}{vol(M)}.$$

A torsion-free sheaf F is called **Mumford-Takemoto stable**, or **slope-stable**, or **μ-stable** if for all subsheaves $F' \subset F$ one has $\operatorname{slope}(F') < \operatorname{slope}(F)$. If F is a direct sum of stable sheaves of the same slope, F is called **polystable**.

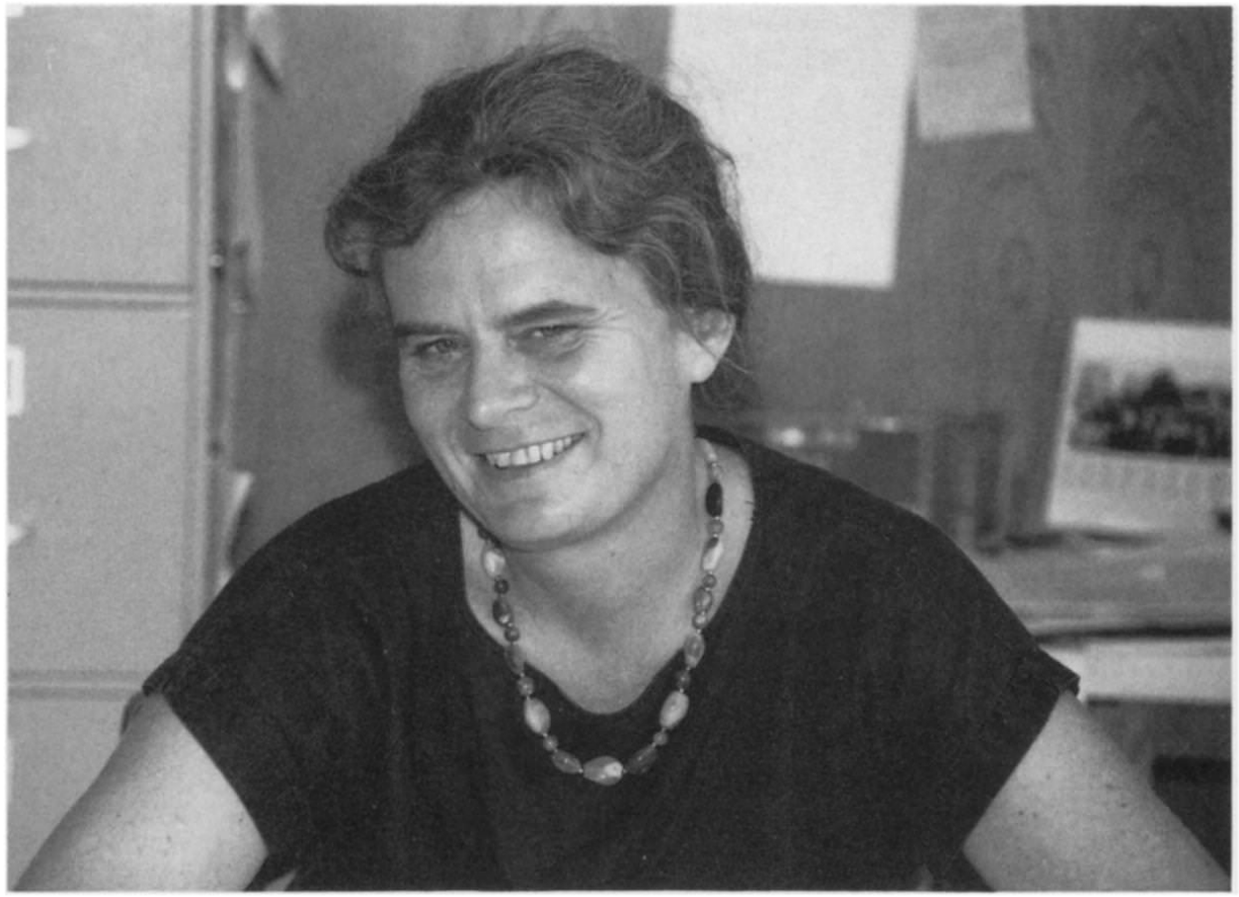
DEFINITION: Consider the operator $\Lambda : \Lambda^1(M) \otimes \operatorname{End} B \longrightarrow \operatorname{End} B$ dual to the multiplication by the Kähler form ω. A Hermitian metric on a holomorphic vector bundle B is called **Yang-Mills** (Hermitian-Einstein) if the curvature of its Chern connection satisfies $\Lambda\Theta_B = \operatorname{slope}(F) \cdot \operatorname{Id}_B$. A Yang-Mills connection is a Chern connection induced by the Yang-Mills metric.

THEOREM: (**Kobayashi-Hitchin correspondence:**)
(Donaldson, Uhlenbeck-Yau). Let B be a holomorphic vector bundle. **Then B admits Yang-Mills connection if and only if B is polystable.** Moreover such a connection is **unique**.

Proof: *Lübke, M.; Teleman, A.:* The Kobayashi-Hitchin correspondence, *World Scientific Publishing Co., 1995.* ■

Sir Simon K. Donaldson

M. Atiyah, N. Hitchin, T. Willmore, S.-T. Yau, Berlin, 1982



Karen Uhlenbeck

The second fundamental form (reminder)

DEFINITION: Let $0 \longrightarrow B_1 \longrightarrow B \longrightarrow B/B_1 \longrightarrow 0$ be an exact sequence of vector bundles, and $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ a connection on B. Recall that **the second fundamental form** of ∇ is the map $A := B_1 \longrightarrow (B/B_1) \otimes \Lambda^1 M$ obtained as a composition of $\nabla\big|_{B_1} : B_1 \longrightarrow B \otimes \Lambda^1 M$ with the projection $B \otimes \Lambda^1 M \longrightarrow (B/B_1) \otimes \Lambda^1 M$.

REMARK: This map **is clearly $C^\infty(M)$-linear.**

The second fundamental form and the curvature

THEOREM 1: Let (B,h) be a Hermitian vector bundle and ∇ an orthogonal connection, $B_1 \subset B$ a sub-bundle, and $B_2 := B_1^\perp$ its orthogonal complement. Denote by $A_1 : B_1 \longrightarrow B_2 \otimes \Lambda^1 M$, $A_2 : B_2 \longrightarrow B_1 \otimes \Lambda^1 M$ the second fundamental forms, and let ∇_1, ∇_2 be connections in B_1, B_2 obtained by applying $\nabla : B_i \longrightarrow B \otimes \Lambda^1 M$ and projecting to $B_i \otimes \Lambda^1 M$. Then

(i) $\nabla = \nabla_1 + \nabla_2 + A_1 + A_2$.

(ii) Consider the second fundamental forms $A_1 \in \Lambda^1 M \otimes \operatorname{Hom}(B_1, B_2)$ and $A_2 \in \Lambda^1 M \otimes \operatorname{Hom}(B_2, B_1)$. **Then for any $x \in TM$, one has $h(A_1(b_1), b_2) = h(b_1, A_2(b_2))$,** in other words, $A_1 = A_2^T$.

(iii) Let $\Lambda^2 M \otimes \operatorname{End}(B) \xrightarrow{\Pi} \Lambda^2 M \otimes \operatorname{End}(B_1)$, be the orthogonal projection and let Θ_{∇_1}, Θ_∇ be the curvature of ∇_1, ∇. **Then $\Theta_{\nabla_1} + A_1 \wedge A_2 = \Pi(\Theta_\nabla)$.**

Proof: (i) is clear, and (iii) follows from (i), because $\Pi(\Theta_\nabla) = \Pi((\nabla_1 + \nabla_2 + A_1 + A_2)^2) = \Pi(\nabla_1^2 + A_1 \wedge A_2)$. The condition (ii) is implied by taking

$$0 = \nabla(h)(b_1, b_2) = d(h(b_1, b_2)) + h(\nabla(b_1), b_2) + h(b_1, \nabla(b_2)) = \\ = h(A_1(b_1), b_2) + h(b_1, A_2(b_2)).$$

■

The second fundamental form of Chern connection

Let now $B_1 \subset B$ be a holomorphic sub-bundle of a holomorphic Hermitian bundle (B,h). Then $\nabla^{0,1}(B_1) \subset B_1 \otimes \Lambda^{0,1}(M)$, **which gives $A_1 \in \Lambda^{1,0}(\operatorname{Hom}(B_1,B_2))$,** where $B_2 = B_1^\perp$. As before, we denote by $\Lambda : \Lambda^2(M) \otimes \operatorname{End}(B) \longrightarrow \operatorname{End}(B)$ the operator adjoint to multiplication by ω.

CLAIM: Let $B_1 \subset B$ be a holomorphic sub-bundle of a holomorphic Hermitian bundle (B,h), $B_2 := B_1^\perp$, and $A_1 \in \Lambda^1 M \otimes \operatorname{Hom}(B_1,B_2)$, $A_2 \in \Lambda^1 M \otimes \operatorname{Hom}(B_2,B_1)$ the corresponding second fundamental forms. **Then $\sqrt{-1}\operatorname{Tr}\Lambda(A_1 \wedge A_2) \geqslant 0$, and the inequality is strict wherever $A_1 \neq 0$.**

Proof: For each $z \in T_m^{1,0}M$, the operator $A_1(z)$ is anti-Hermitian and adjoint to $A_2(\overline{z})$ because $h(A_1(b_1),b_2) = h(b_1,A_2(b_2))$. Therefore, the operator $A := A_1 \wedge A_2(z,\overline{z})$ is a square of an anti-Hermitian operator. **The square of an anti-Hermitian operator R is a Hermitian operator with negative eigenvalues:** $h(R(b),R(b)) = -h(R^2(b),b) \geqslant 0$. Since $\Lambda(A_1 \wedge A_2) = \sum_i A_1 \wedge A_2(z_i,\overline{z}_i)$ for some orthonormal basis $z_1,...,z_n \in T_m^{1,0}M$, and each of these terms is either zero or sign-definite of the same sign, this implies $\sqrt{-1}\operatorname{Tr}\Lambda(A_1 \wedge A_2) = \sqrt{-1}\sum_i A_1 \wedge A_2(z_i,\overline{z}_i) \geqslant 0$, with equality at $m \in M$ only when $A_1\big|_{T_mM} = 0$. ■

Kobayashi-Hitchin correspondence (the easy part)

COROLLARY: Let $B_1 \subset B$ be a holomorphic sub-bundle of a holomorphic Hermitian bundle (B, h), and $\Theta_{\nabla_1} + A_1 \wedge A_2 = \Pi(\Theta_\nabla)$ the decomposition of curvature obtained in Theorem 1 (iii). **Then $\sqrt{-1}\operatorname{Tr}(\Theta_{\nabla_1} - \Pi(\Theta_\nabla))(x, Ix) \geqslant 0$,** for any tangent vector $x \in T_m M$. Moreover, **if $A_1(x) \neq 0$, this inequality is strict.** ■

REMARK: Recall that a connecion ∇ on a vector bundle B over a Kähler manifold **is Yang-Mills if and only if ∇ is Hermitian, and its curvature satisfies $\Theta_\nabla \in \Lambda^{1,1}(M) \otimes \operatorname{End}(B)$ and $\Lambda(\Theta_\nabla) = const \cdot \operatorname{Id}_B$.**

DEFINITION: A vector bundle (B, ∇) with connection is called **undecomposable** if (B, ∇) is not isomorphic to $(B_1, \nabla_1) \oplus (B_2, \nabla_1)$.

THEOREM: Let ∇ be a Yang-Mills connection on a vector bundle B over a Kähler manifold, and $B_1 \subset B$ its holomorphic sub-bundle. Assume that B is undecomposable. **Then $\operatorname{slope}(B_1) < \operatorname{slope}(B)$.**

Proof: Next slide.

Kobayashi-Hitchin correspondence (the easy part) (2)

Proof. Step 1: Since ∇ is Yang-Mills, we have $\Lambda\Theta_\nabla = r\,\mathrm{Id}_B$. The constant r can be recovered from the Chern class as follows. Let $s := \operatorname{rk} B$ and $s_1 := \operatorname{rk} B_1$. By definition of Λ we have $\int_M c_1(B)\wedge\omega^{n-1} = \int_M \operatorname{Tr}(\Lambda(\Theta_\nabla))\omega^n = rs\int_M \omega^n$, hence $rs = \frac{\int_M c_1(B)\wedge\omega^{n-1}}{\int_M \omega^n}$. **This gives $r = \operatorname{slope} B$.**

Step 2: Since B is indecomposable, the second fundamental form of B_1 is non-zero. Denote by Π the orthogonal projection $\Lambda^2 M\otimes\operatorname{End}(B) \xrightarrow{\Pi} \Lambda^2 M\otimes\operatorname{End}(B_1)$. **Applying $\operatorname{Tr}_B\Lambda$ to both sides of Theorem 1 (iii), we obtain $\operatorname{Tr}_B\Lambda(\Pi(\Theta_\nabla)) = \operatorname{Tr}_B\Lambda(\Theta_{\nabla_1}) - \operatorname{Tr}_B\Lambda(A_1\wedge A_1^T)$ $(*)$.**

Step 3: By construction, Π commutes with Λ, hence $\Lambda(\Pi(\Theta_\nabla)) = r\,\mathrm{Id}_{B_1}$. Denote by V the volume $V := \int_M \omega^n$. Then $V^{-1}\int_M \operatorname{Tr}\Pi(\Theta_\nabla)\omega^{n-1} = rs_1 = s_1\operatorname{slope}(B)$. Multiplying (*) by V^{-1} and integrating, we obtain

$$\begin{aligned} s_1\operatorname{slope}(B_1) &= V^{-1}\int_M c_1(B_1)\wedge\omega^{n-1} = \\ &= V^{-1}\int_M \operatorname{Tr}_B\Lambda(\Pi(\Theta_\nabla))\omega^n - V^{-1}\int_M \operatorname{Tr}_B\Lambda(A_1\wedge A_1^T)\omega^n = \\ &= rs_1 - \int_M \operatorname{Tr}_B\Lambda(A_1\wedge A_1^T)\omega^n < s_1\operatorname{slope}(B), \end{aligned}$$

giving $\operatorname{slope}(B_1) < \operatorname{slope}(B)$. ■

The Bogomolov-Lübke inequality

DEFINITION: Note that the form $\operatorname{Tr}(\Theta \wedge \Theta)$ represents the cohomology class $c_2(B) - \frac{r-1}{r} c_1(B)^2$, where Θ is a curvature of a connection in a complex vector bundle, and r its rank. The class $c_2(B) - \frac{r-1}{r} c_1(B)^2$ is called **the discriminant** of B.

THEOREM: Let B be a stable bundle over a compact Kähler manifold M, with $c_1(B) = c_2(B) = 0$, and ∇ its Yang-Mills connection. **Then ∇ is flat.**

This theorem is a special case of the following result.

THEOREM: Let B be a stable bundle over a compact Kähler manifold M, ∇ its Yang-Mills connection, and θ its curvature. Assume that $\operatorname{slope}(B) = 0$. Denote by $D \in H^4(M)$ the discriminant of B, $D = c_2(B) - \frac{r-1}{r} c_1(B)^2$. **Then $\int_M D \wedge \omega^{n-2} \geqslant 0$, and the inequality may happen only when $\Theta = 0$.**

REMARK: The condition $\operatorname{slope}(B) = 0$ is used to simplify the statement, **a similar result can be stated for any slope.**

The Bogomolov-Lübke inequality (proof)

THEOREM: Let B be a stable bundle over a compact Kähler manifold M, ∇ its Yang-Mills connection, and Θ its curvature. Assume that $\operatorname{slope}(B) = 0$. Denote by $D \in H^4(M)$ the discriminant of B, $D = c_2(B) - \frac{r-1}{r} c_1(B)^2$. **Then $\int_M D \wedge \omega^{n-2} \geqslant 0$, and the inequality may happen only when $\Theta = 0$.**

Proof. Step 1: Since the connection ∇ is Hermitian, it preserves the natural real structure in $\Lambda^{1,1}(M) \otimes \operatorname{End}(B)$, $\eta \otimes b \longrightarrow \overline{\eta} \otimes b^\perp$, where by $b^\perp$ one understands the Hermitian adjoint endomorphism. Therefore, **we may assume that Θ is real with respect to this real structure.**

Step 2: Let θ, $\theta_1, ..., \theta_n$ be an orthonormal basis in $\Lambda^{1,0}(M)$. Consider the decomposition $\Theta = \sum_{i \neq j} (\theta_i \wedge \overline{\theta}_j + \overline{\theta}_i \wedge \overline{\theta}_j) \otimes b_{ij} + \sum_i (\theta_i \wedge \overline{\theta}_i) \otimes a_i$ with $b_{ij}, a_i \in \mathfrak{u}(B)$. Let $\Xi := \operatorname{Tr}(\Theta \wedge \Theta)$. Then $\Xi \wedge \omega^{n-2} = const \operatorname{Tr}\left(-\sum_{i \neq j} b_{ij}^2 + \sum_{i \neq j} a_i a_j\right)$, where $const$ is a positive constant. On the other hand, $\Lambda(\Theta) = \sum a_i = 0$, which brings $\sum_{i<j} a_i a_j = -\sum_i a_i^2$. This gives

$$\Xi \wedge \omega^{n-2} = const \operatorname{Tr}\left(-\sum_{i \neq j} b_{ij}^2 - \sum_i a_i^2\right).$$

Since the Killing form on $\mathfrak{u}(B)$ is negative definite, this sum is positive, and strictly positive unless $\Xi = 0$. ■