Hyperkähler manifolds,

lecture 23: the period map and the BBF form

IMPA, sala 236

Misha Verbitsky, June 23, 2023, 13:30

http://verbit.ru/IMPA/HK-2023/

Teichmüller space

DEFINITION: The space of almost complex structures is an infinite-dimensional Fréchet manifold X_M of all tensors $I^2 = -\operatorname{Id}_{TM}$, equipped with the natural Fréchet topology.

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the space of complex structures on M, and let Teich := Comp / $\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

REMARK: The space of $\operatorname{Diff}_0(M)$-orbits in a small neighbourhood of a point in Comp is always **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady). However, the quotient Comp / $\operatorname{Diff}_0(M)$ **is often non-Hausdorff**.

DEFINITION: We call $\Gamma := \operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ **the mapping class group**.

REMARK: The topology of the space Teich $/\Gamma$ is often bizzarre. However, **its points are in bijective correspondence with equivalence classes of complex structures.**

Deformations of holomorphically symplectic manifolds.

THEOREM: (Kodaira-Spencer) A small deformation of a compact Kähler manifold is again Kähler.

COROLLARY: A small deformation of a holomorphically symplectic Kähler manifold M **is again holomorphically symplectic and Kähler.**

Proof: A small deformation M' of M would satisfy $H^{2,0}(M') = H^{2,0}(M)$, by semi-continuity of Hodge numbers; however, a small deformation of a non-degenerate (2,0)-form remains non-degenerate. ∎

DEFINITION: A compact complex manifold admitting holomorphically symplectic and Kähler structure is called **a manifold of hyperkähler type**

REMARK: From now on, Teich **denotes the Teichmüller space of complex structures of hyperkähler type.** It is an open subset in the Teichmüller space of complex structures.

The period map and Bogomolov's local Torelli theorem

Definition: Let $\operatorname{Per}: \operatorname{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map J to a point $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega \rangle \in \operatorname{Gr}(2, H^2(M, \mathbb{R}))$ The map $\operatorname{Per}: \operatorname{Teich} \longrightarrow \operatorname{Gr}(2, H^2(M, \mathbb{R}))$ is called **the period map**.

THEOREM: (Bogomolov's local Torelli theorem)
Let M be a maximal holonomy hyperkähler manifold, and Teich its Teichmüller space. The the period map $\operatorname{Per}: \operatorname{Teich} \longrightarrow \operatorname{Gr}(2, H^2(M, \mathbb{R}))$ **is locally a diffeomorphism.**

REMARK: Bogomolov's theorem implies that Teich **is smooth.** It is **non-Hausdorff** even in the simplest examples.

Today I will assume Bogomolov's theorem. **I will deduce from Bogomolov's theorem a result about topology of hyperkähler manifolds of maximal holonomy.**

Polynomial invariants of Lie groups

PROPOSITION 1: Let V be a real vector space equipped with an action of a Lie group G, and Q a G-invariant polynomial function. Let $S \subset \operatorname{Gr}(2, V)$ be an open subset in the Grassmannian of 2-planes. Assume that for any $W \in S$, there exists a subgroup $\rho_W \subset G$ isomorphic to S^1 acting by rotations on W and trivially on V/W. **Then Q is proportional to q^n, where q is a quadratic form on V.**

Proof. Step 1: Let $W \in S$ be a 2-plane in V. **Any rotation-invariant polynomial function on $\mathbb{R}^2$ is a power of quadratic form (prove this as an exercise)**, hence $Q|_W = \lambda q_W^n|_W$, for some quadratic form q_W.

Step 2: We want to take the n-th root of Q. When n is odd, the n-th root of Q is well defined. When n is even, the restriction $Q|_W$ does not change sign, hence Q does not change sign on the set $U_S \subset V$ of all vectors passing through planes $W \in S$. **The function $q := \sqrt[n]{\pm Q}$ is well defined on the whole of V when n is odd, and on an open subset $U_S \subset V$ when it is even.**

Polynomial invariants of Lie groups (2)

PROPOSITION 1: Let V be a real vector space equipped with an action of a Lie group G, and Q a G-invariant polynomial function. Let $S\subset \operatorname{Gr}(2,V)$ be an open subset in the Grassmannian of 2-planes. Assume that for any $W\in S$, there exists a subgroup $\rho_W\subset G$ isomorphic to S^1 acting by rotations on W and trivially on V/W. **Then Q is proportional to q^n, where q is a quadratic form on V.**

Step 2: The function $q:=\sqrt[n]{\pm Q}$ is well defined on the whole of V when n is odd, and on an open subset $U_S\subset V$ when it is even.

Step 3: The function $q: U_S \longrightarrow \mathbb{R}$ is a polynomial of second degree on all hyperplanes $W\in S$. Consider the second derivative $H:=\frac{d^2}{dxdy}q$ as a section of $\operatorname{Sym}^2 T^*U_S$. Take $\zeta\in T_vU_S=V$ such that $\langle\zeta,v\rangle\in S$. Since q is a quadratic function on $\langle\zeta,v\rangle$, the value of the function $v\mapsto H(\zeta,\zeta)$ is independent from v. The set of ζ for which this is true is open, and $H(\zeta,\zeta)$ is a quadratic polynomial on ζ. **This implies that $v\longrightarrow H(\zeta,\zeta)$ is constant on U_S, for any $\zeta\in V$.**

Step 4: A function which satisfies $\frac{d^2}{dxdy}q=const$ is a quadratic polynomial. We extend it to a quadratic polynomial on V. **Then $Q=\lambda q^n$ on U_S.** Since Q is polynomial, and $U_S\subset V$ is open, this expression is true everywhere. ■

The image of the period map is open

The following theorem is the most general version of local Torelli which might be used to define the BBF form.

We might prove it at some later date (or not).

PROPOSITION: Let M be a compact holomorphically symplectic manifold **(not necessarily Kähler)** such that $H^{0,2}(M) = H^{2,0}(M) = \mathbb{C}$, and assume that all ∂-exact holomorphic 3-forms on M vanish. **Then the period map has an open image in** $\operatorname{Gr}(2, H^2(M, \mathbb{R}))$**.**

REMARK: These assumptions **are clearly true when M is a compact hyperkähler manifold of maximal holonomy.** Indeed, **on a compact Kähler manifold all exact holomorphic forms vanish.**

The BBF form

THEOREM: Let M be a compact holomorphically symplectic manifold, $\dim_{\mathbb{C}} M = 2n$, admitting the Hodge decomposition on $H^2(M)$. Assume that all ∂-exact holomorphic 3-forms on M vanish, and $H^{0,2}(M) = H^{2,0}(M) = \mathbb{C}$. **Then the space $H^2(M)$ is equipped with a bilinear symmetric form q such that for any $\eta \in H^2(M)$, one has $\int_M \eta^{2n} = q(\eta, \eta)^n$.**

Proof. Step 1: Consider the Hodge decomposition on $H^2(M)$ induced by the complex structure $I \in U$. This gives **"the Hodge rotation map"**, that is, an $U(1)$-action $\rho_I(t)$, acting as $e^{2\pi\sqrt{-1}\,(p-q)t}$ on $H^{p,q}(M)$. **Clearly, the polynomial $Q(\eta) := \int_M \eta^{2n}$ is ρ_I-invariant.** By definition, ρ_I acts trivially on $H^{1,1}(M)$ and rotates $W = \langle \operatorname{Re}\Omega, \operatorname{Im}\Omega\rangle$.

Step 2: Let G be the Lie group generated by the Hodge rotation maps ρ_I for all complex structures I satisfying the assumptions of the theorem. Since the image of the period map is open, **the action of G satisfies assumptions of Proposition 1, giving $Q(\eta) = \lambda q(\eta, \eta)^n$.** ■

DEFINITION: Usually one normalizes q in such a way that it is integer and primitive; then $Q(\eta) = \lambda q(\eta, \eta)^n$, where $\lambda > 0$ is called **the Fujiki constant**. The form q is called **the Bogomolov-Beauville-Fujiki form** (the BBF form).

Further directions

We can finish the course by proving the local Torelli theorem (and state the global Torelli), or the following result

THEOREM: Let M be a hyperkähler manifold of maximal holonomy, and $\mathfrak{g} \subset \operatorname{End}(H^*(M))$ the Lie algebra generated by all Lefschetz $\mathfrak{sl}(2)$-triples, for all Kähler structures of hyperkähler type. **Then $\mathfrak{g} = \mathfrak{so}(4, b_2 - 2)$.** Moreover, **the subalgebra A^* of $H^*(M, \mathbb{Q})$ generated multiplicatively by $H^2(M)$ satisfies** $A^{2k} = \operatorname{Sym}^k(H^*(M, \mathbb{Q}))$ for any $k \leqslant \frac{1}{2} \dim_{\mathbb{C}} M$.

Your choice!

Kodaira-Spencer stability theorem.

THEOREM: (Kodaira-Spencer) Consider a smooth family I_t of complex structures on a compact manifold M, $t \in]-a,a[$. Assume that (M, I_0) admits Kähler structure. **Then there exists a neighbourhood $W \ni 0$ such that for each $t \in W$, the manifold (M, I_t) is Kähler.**

Proof. Step 1: We denote (M, I_t) by X_t and the corresponding family of complex manifolds over $B :=]-a,a[$ by $\mathcal{X}$. Consider the relative Frölicher spectral sequence

$$R^i\pi_*(\Omega^j_B\mathcal{X}) \Rightarrow R^{i+j}\pi_*(\mathbb{C}_{\mathcal{X}}) \quad (*)$$

where $\Omega^j_B\mathcal{X}$ is fiberwise holomorphic forms on the fibers. Here $R^{i+j}\pi_*(\mathbb{C}_{\mathcal{X}})$ is the derived pushforward of a constant sheaf (that is, a graded local system over B with the fibers of grading k in $y \in B$ identified with k-th cohomology of X_y).

It is a relative (over B) version of the usual Frölicher spectral sequence $H^i(\Omega^j M) \Rightarrow H^{i+j}(M, \mathbb{C})$. **This spectral sequence gives an inequality**

$$\sum_{i+j=k} \dim H^i(\Omega^j X_0) \geqslant \sum_{i+j=k} \dim H^i(\Omega^j X_z) \quad (**)$$

for general $z \in B$ because the cohomology of $\Omega^j X_t$ are semicontinuous in t.

Kodaira stability theorem (part 2)

Proof. Step 1: This spectral sequence gives an inequality

$$\sum_{i+j=k} \dim H^i(\Omega^j X_0) \geqslant \sum_{i+j=k} \dim H^i(\Omega^j X_z) \quad (**)$$

for general $z \in B$ because the cohomology of coherent sheaves on X_t are semicontinuous in t.

Proof. Step 2: Since X_0 is Kähler, the Frölicher spectral sequence for X_0 degenerates in E_2, giving $\sum_{i+j=k} \dim H^i(\Omega^j X_z) = h^k(X_z)$. By semicontinuity,

$$\sum_{i+j=k} \dim H^i(\Omega^j X_y) \leqslant \sum_{i+j=k} \dim H^i(\Omega^j X_0)$$

in a sufficiently small neighbourhood U of $0 \in B$. Comparing this with (**), we find that **rank of $H^i(\Omega^j X_y)$ is constant in U,** hence the inequality (**) is equality in U, and **the spectral sequence (*) degenerates.**

Kodaira stability theorem (part 3)

Step 3: Consider the sheaf $\mathcal{H} := R^1\pi_*(\Omega^1_B\mathcal{X})$. By Step 2, $\mathcal{H}$ is a vector bundle in a neighbourhood of 0, generated by fiberwise $\overline{\partial}$-closed (1,1)-forms. up to fiberwise $\overline{\partial}$-exact (1,1)-forms. Let $\Lambda^{1,1}_{cl}(\mathcal{X}/B)$ be the sheaf of fiberwise closed fiberwise forms on $\mathcal{X}$, and $\pi_*\Lambda^{1,1}_{cl}(\mathcal{X}/B) \xrightarrow{\Xi} \mathcal{H}$ the natural projection. Choose a Hermitian metric on $\mathcal{X}$, smoothly extending the Kähler metric ω_z on X_z, and let $\mathcal{H} \xrightarrow{\Xi^*} \pi_*\Lambda^{1,1}_{cl}(\mathcal{X}/B)$ be the Hermitian adjoint map. By construction, Ξ^* is an orthogonal projection of cohomology to closed (1,1)-forms along the exact 2-forms. Therefore, **Ξ^* maps the Kähler class $[\omega_z]$ to its harmonic representative ω_z.**

Step 4: Let $\tilde{\omega}$ be a smooth section of $\mathcal{H}$ satisfying $\tilde{\omega}|_z = [\omega_z]$. **Then $\Xi^*(\tilde{\omega})$ is a family of closed forms $\omega_y \in \Lambda^{1,1}_{cl}(X_y)$, depending smoothly on $y \in B$.** Since all eigenvalues of ω_z are positive, the same is true for ω_y for y sufficiently close to z. However, a closed, positive $(1,1)$-form is Kähler. ■