Hyperkähler manifolds,

lecture 24: Fujiki formula and its applications

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The BBF form (reminder)

THEOREM: Let M be a compact holomorphically symplectic manifold, $\dim_{\mathbb C} M = 2n$, admitting the Hodge decomposition on $H^2(M)$. Assume that all ∂-exact holomorphic 3-forms on M vanish, and $H^{0,2}(M) = H^{2,0}(M) = \mathbb{C}$. **Then the space $H^2(M)$ is equipped with a bilinear symmetric form q such that for any $\eta \in H^2(M)$, one has $\int_M \eta^{2n} = q(\eta,\eta)^n$.**

Proof. Step 1: Consider the Hodge decomposition on $H^2(M)$ induced by the complex structure $I \in U$. This gives **"the Hodge rotation map"**, that is, an $U(1)$-action $\rho_I(t)$, acting as $e^{2\pi\sqrt{-1}\,(p-q)t}$ on $H^{p,q}(M)$. **Clearly, the polynomial $Q(\eta) := \int_M \eta^{2n}$ is ρ_I-invariant.** By definition, ρ_I acts trivially on $H^{1,1}(M)$ and rotates $W = \langle \operatorname{Re}\Omega, \operatorname{Im}\Omega\rangle$.

Step 2: Let G be the Lie group generated by the Hodge rotation maps ρ_I for all complex structures I satisfying the assumptions of the theorem. Since the image of the period map is open, **the action of G satisfies assumptions of Proposition 1, Lecture 23, giving $Q(\eta) = \lambda q(\eta,\eta)^n$.** ■

DEFINITION: Usually one normalizes q in such a way that it is integer and primitive; then $Q(\eta) = \lambda q(\eta,\eta)^n$, where $\lambda > 0$ is called **the Fujiki constant**. The form q is called **the Bogomolov-Beauville-Fujiki form** (the BBF form).

Hodge decomposition on hk manifolds (reminder from Lecture 21)

EXERCISE: (Klebsch-Gordan formula)
Let V_p and V_q, $p \geqslant q$ be a $(p+1)-$ and $(q+1)$-dimensional complex irreducible representation of $SU(2)$. **Prove that $V_p \otimes V_q = \bigoplus_{i=0}^{q} V_{p+q-2i}$.**

COROLLARY: Let M be a hyperkähler manifold, $\dim_{\mathbb H} M = n$. Then for each $x \in M$, and each $d \leqslant 2n$, **the space $\Lambda^d_x(M)$ is a direct sum of several V_i for $0 \leqslant i \leqslant d$,** of the same parity as d, and the same holds for $H^d(M)$.

Proof: Follows from Klebsch-Gordan because $\Lambda^1_x M$ is a direct sum of $SU(2)$-representations of weight 1, $\Lambda^1_x M = V_1^n$. ∎

CLAIM: Let M be a compact hyperkähler manifold of maximal holonomy. **Then $H^2(M) = \langle \omega_I, \omega_J, \omega_K\rangle \oplus H^2(M)^{SU(2)}$,** where $H^2(M)^{SU(2)}$ is the space of $SU(2)$-invariant vectors in $H^2(M)$.

Proof: A class $\eta \in H^2(M)$ is $SU(2)$-invariant if and only if η is of Hodge type $(1,1)$ with respect to all induced complex structures. However, $\dim H^{2,0}(M,I) = 1$, hence the sub-representation W generated by $H^{2,0}(M,I)$ is 1-dimensional. It is a subrepresentation of weight 2, hence it is 3-dimensional, hence $W = \langle \omega_I, \omega_J, \omega_K\rangle$. Its complement is an $SU(2)$-invariant subspace $V \subset H^{1,1}_I(M)$; by $SU(2)$-invariance, this gives $V \subset H^{1,1}_L(M)$ for all induced complex structures, hence $V = H^2(M)^{SU(2)}$. ∎

Primitive cohomology classes

Recall that a cohomology class $\eta \in H^2(M)$ on Kähler manifold (M,ω) is called **primitive** if $\int_M \eta \wedge \omega^{\dim_{\mathbb C} M-1} = 0$. The following result is a special case of **"Hodge-Riemann relations".**

PROPOSITION: (Hodge index theorem)
Let $\eta \in H^{1,1}(M,\mathbb{R})$ be a non-zero primitive $(1,1)$-class on a compact Kähler manifold (M,ω). **Then $\int_M \eta \wedge \eta \wedge \omega^{\dim_{\mathbb C} M-2} < 0$.**

Proof: Take a harmonic representative for η. Since $\Lambda\eta = 0$, $\eta \perp \omega$ at each point of M. Consider a $2n$-dimensional real vector space V equipped with a Hermitian structure (I,g,ω), and let $\eta \in \Lambda^{1,1}(V)$ be a form which is orthogonal to ω. **It would suffice to show that $\eta\wedge\eta\wedge\omega^{\dim_{\mathbb C} V-2}$ is a strictly negative volume form on V.**

LEMMA: Let (V,I,g,ω) be a real vector space equipped with a Hermitian structure, and $\eta \in \Lambda^{1,1}(V)$ a non-zero form which is orthogonal to ω. **Then $\frac{\eta\wedge\eta\wedge\omega^{\dim_{\mathbb C} V-2}}{\operatorname{Vol}_V} < 0$.**

Proof: Next slide

Hodge index theorem for pseudo-Hermitian 2-forms on $\mathbb{R}^{2n}$

LEMMA: Let (V, I, g, ω) be a real vector space equipped with a Hermitian structure, and $\eta \in \Lambda^{1,1}(V)$ a non-zero form which is orthogonal to ω. **Then** $\frac{\eta\wedge\eta\wedge\omega^{\dim_{\mathbb{C}} V-2}}{\operatorname{Vol}_V} < 0$.

Step 1: Let $h(x,y) := \eta(x, Iy)$. Since η is of type $(1,1)$, **the form h is I-invariant, that is, pseudo-Hermitian.** For any pseudo-Hermitian form on a Hermitian space, there exists a basis where both forms are diagonal. Therefore, **there exists an orthonormal basis $x_1, .., x_n, y_1, ..., y_n$ in V^* such that $I(x_i) = y_i$, $I(y_i) = -x_i$ and $h = \sum_{i=1}^n u_i(x_i \otimes x_i + y_i \otimes y_i)$.**

Step 2: The form g is written in the same basis: $g = \sum_{i=1}^n x_i \otimes x_i + y_i \otimes y_i$ Therefore, the scalar product (h, g) is equal to $2\sum_{i=1}^n u_i$, and $(\omega, \eta) = 0$ is equivalent to $\sum_{i=1}^n u_i = 0$. Clearly, $\eta\wedge\eta\wedge\omega^{\dim_{\mathbb{C}} V-2} = \frac{1}{n(n-1)} \sum_{i<j} u_i u_j \omega^n$. Since $\sum_{i=1}^n u_i = 0$, we have

$$0 = \left(\sum_{i=1}^n u_i\right)^2 = \sum_{i=1}^n u_i^2 + 2\sum_{i<j} u_i u_j,$$

giving $\sum_{i<j} u_i u_j < 0$. ■

Fujiki formula

REMARK: For any Kähler form, $\int_M \omega^{\dim_{\mathbb{C}} M} > 0$, hence the sign of $q(\omega,\omega)$ is constant when ω runs through the Kähler cone. If the choice of the sign in BBF form is ambiguous, **we fix the sign of q in such a way that $q(\omega,\omega) > 0$ for Kähler ω.**

CLAIM: The BBF form is $SU(2)$-invariant, where the $SU(2)$-action is induced by any hyperkähler structure.

Proof: Indeed, the $SU(2)$-action is generated by the Hodge rotation ρ_L, where L are induced complex structures. ■

PROPOSITION: (Fujiki formula)
Let $\eta_1, ..., \eta_{2n}$ be a collection of classes in $H^2(M, \mathbb{R})$. Denote by Σ_{2n} the symmetric group, with a permutation $\sigma = (\sigma_1, ..., \sigma_{2n})$ taking $(1, ..., 2n)$ to $\sigma_1, ..., \sigma_{2n}$ **Then**

$$\int_M \eta_1 \wedge ... \wedge \eta_{2n} = A \sum_{\sigma \in \Sigma_{2n}} q(\eta_{\sigma_1}, \eta_{\sigma_2}) q(\eta_{\sigma_3}, \eta_{\sigma_4}) ... q(\eta_{\sigma_{2n-1}}, \eta_{\sigma_{2n}}),$$

where A is a positive rational constant.

Proof: For any degree p homogeneous polynomial function $f \in \operatorname{Sym}^p V^*$, its **polarization** is a symmetric form $Q := \frac{d^p f}{d\zeta_1 d\zeta_2 ... d\zeta_p} f \in (V^*)^{\otimes p}$, satisfying $Q(\zeta, \zeta, ..., \zeta) = f(\zeta)$. The Fujiki formula is obtained by applying the polarization to both sides of $\int_M \eta^{2n} = \lambda q(\eta,\eta)^n$. ■

Primitive cohomology classes on hyperkähler manifolds

CLAIM: Let M be a hyperkähler manifold of maximal holonomy. Then **the space $H^{1,1}_{prim}(M)$ of primitive $(1,1)$-classes is $H^2(M)^{SU(2)}$**.

Proof: Primitive classes are represented by forms in $\ker\Lambda$, where Λ is the Hodge operator. Therefore, a harmonic form is primitive if and only if it is orthogonal to ω pointwise. The decomposition $H^2(M) = \langle\omega_I, \omega_J, \omega_K\rangle \oplus H^2(M)^{SU(2)}$ is orthogonal, hence the decomposition $H^{1,1}_I(M) = \mathbb{R}\omega_I \oplus H^2(M)^{SU(2)}$ is also orthogonal, giving $H^2(M)^{SU(2)} = H^{1,1}_{prim}(M)$. ■

Corollary 1: Let M be a hyperkähler manifold of maximal holonomy, $\dim_{\mathbb{C}} M = 2n$, and $\eta \in H^{1,1}(M)$ primitive. **Then $q(\eta,\omega) = 0$ and $q(\eta,\eta)$ is negative.**

Proof. Step 1: $0 = \int_M \eta\wedge\omega^{\dim_{\mathbb{C}} M-1} = \lambda q(\eta,\omega)q(\omega,\omega)^{n-1}$. Since $q(\omega,\omega) > 0$, this gives $q(\eta,\omega) = 0$.

Step 2: By Hodge index theorem, $\int_M \eta\wedge\eta\wedge\omega^{\dim_{\mathbb{C}} V-2} < 0$. Fujiki formula gives $\int_M \eta\wedge\eta\wedge\omega^{\dim_{\mathbb{C}} V-2} = c_1 q(\eta,\omega)^2 q(\omega,\omega)^{n-2} + c_2 q(\eta,\eta)q(\omega,\omega)^{n-1}$, where c_1, c_2 are positive constants. Since $q(\eta,\omega) = 0$ (Step 1), $\int_M \eta\wedge\eta\wedge\omega^{\dim_{\mathbb{C}} V-2} < 0$ gives $q(\eta,\eta)q(\omega,\omega)^{n-1} < 0$, hence $q(\eta,\eta) < 0$. ■

Signature of the BBF form

PROPOSITION: Let M be a hyperkähler manifold of maximal holonomy, and $q \in \operatorname{Sym}^2(H^2(M,\mathbb{Q})^*)$ the BBF form. **Then q is a non-degenerate form of signature $(3, b_2 - 3)$.** It is positive on the subspace in $H^2(M,\mathbb{R})$ generated by $\omega_I, \omega_J, \omega_K$, and negative definite on the space $H^2(M,\mathbb{R})^{SU(2)}$ of $SU(2)$-invariant classes.

Proof. Step 1: As we have already seen, $H^2(M,\mathbb{R}) = H^2(M,\mathbb{R})^{SU(2)} \oplus \langle\omega_I, \omega_J, \omega_K\rangle$. The form q on $\langle\omega_I, \omega_J, \omega_K\rangle$ is positive definite, because it is positive on ω_I and $SU(2)$-invariant. **It remains only to show that q is negative definite on $H^2(M,\mathbb{R})^{SU(2)}$.**

Step 2: The space $H^2(M,\mathbb{R})^{SU(2)}$ coincides with the space $H^{1,1}_{prim}(M)$ of primitive (1,1)-forms, and q **is negative definite on $H^{1,1}_{prim}(M)$ by Corollary 1.** ■

Beauville-Bogomolov formula

THEOREM: (**Beauville-Bogomolov formula**)
Let Ω be a holomorphic symplectic form on a hyperkähler manifold of maximal holonomy. **Then the BBF form is equal to**

$$q'(\eta,\eta') := \lambda \int_M \eta \wedge \eta' \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \\ - R\lambda \frac{\left(\int_M \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^n\right) \cdot \left(\int_M \eta' \wedge \Omega^n \wedge \overline{\Omega}^{n-1}\right)}{\int_M \Omega^n \wedge \overline{\Omega}^n}, \qquad (*)$$

where $R = \frac{2(n-1)}{n}$, and λ is a positive constant.

Proof. Step 1: The precise value of R is irrelevant and is left as an exercise. Since q is compatible with the Hodge decomposition, we have $q(\eta,\Omega) = q(\eta,\overline{\Omega}) = q(\Omega,\Omega) = q(\overline{\Omega},\overline{\Omega}) = 0$ for any $\eta \in H^{1,1}(M)$. By Fujiki formula, this gives $q(\eta,\eta)q(\Omega,\overline{\Omega})^{n-2} = const \int_M \eta^2 \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1}$. Therefore, the form q' is proportional to q on the space $H^{1,1}(M)$. **We fix the coefficients λ and $R\lambda$ in such a way that $q(x,\overline{x}) = q'(x,\overline{x})$ when $x \in H^{1,1}(M)$ and $x = \Omega$.** It remains to show that $q = q'$ for such choice of constants.

Beauville-Bogomolov formula (2)

THEOREM: The BBF form is equal to

$$q'(\eta,\eta') := \lambda \int_M \eta\wedge\eta'\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ - R\lambda \frac{\left(\int_M \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\cdot\left(\int_M \eta'\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)}{\int_M \Omega^n\wedge\overline{\Omega}^n}, \qquad (*)$$

Proof. Step 1: ...We fix the coefficients λ and $R\lambda$ in such a way that $q(x,\overline{x}) = q'(x,\overline{x})$ when $x\in H^{1,1}(M)$ and $x=\Omega$... It remains to show that $q=q'$ for such choice of constants.

Step 2: By construction, the forms q and q' are equal on $H^{1,1}(M)$. Since both forms are compatible with the Hodge decomposition, they satisfy $q(\eta,\Omega) = q(\eta,\overline{\Omega}) = q(\Omega,\Omega) = q(\overline{\Omega},\overline{\Omega}) = 0$ and $q'(\eta,\Omega) = q'(\eta,\overline{\Omega}) = q'(\Omega,\Omega) = q'(\overline{\Omega},\overline{\Omega}) = 0$. Therefore, $q=q'$ is implied by $q(\Omega,\overline{\Omega}) = q'(\Omega,\overline{\Omega})$ and $q(\eta,\eta) = q'(\eta,\eta)$. ■

REMARK: A similar argument proves the following formula (which we won't be using)

$$q(\eta,\eta') = \lambda\int_X \omega^{n-2}\wedge\eta\wedge\eta' - \lambda\frac{n-2}{(n-1)^2}\cdot\frac{\int_X\omega^{n-1}\eta\cdot\int_X\omega^{n-1}\eta'}{\int_X\omega^n}$$

The Lie group generated by Hodge rotations

Let (M, I) be a compact Kähler manifold. Recall that **the Hodge rotation** is $U(1)$-action $\rho_I(t) \subset \operatorname{Aut}(H^*(M, \mathbb{R}))$, acting as $e^{2\pi\sqrt{-1}\,(p-q)t}$ on $H^{p,q}(M)$. Recall that $O(p, q)$, $p, q > 0$ has 4 connected components. **The connected component of unity is denoted $SO^+(H^2(M, \mathbb{R}), q)$.**

THEOREM: Let $G \subset \operatorname{Aut}(H^*(M, \mathbb{R}))$ be the subgroup generated by all Hodge rotations $\rho_I(t)$, for all complex structures I of hyperkähler type. **Then G acts on $H^2(M, \mathbb{R})$ as $SO^+(H^2(M, \mathbb{R}), q)$.**

Proof. Step 1: The image G' of G in $GL(H^2(M, \mathbb{R}))$ is a connected subgroup preserving $\pm q$, hence $G' := G\big|_{H^2(M,\mathbb{R})}$ belongs to $SO^+(H^2(M, \mathbb{R}), q)$.

Step 2: This group contains all Hodge rotations, which can be identified with rotations in a 2-dimension subspace $V \in \mathbb{P}\mathrm{er} \subset \operatorname{Gr}(2, H^2(M, \mathbb{R}))$. Since $\mathbb{P}\mathrm{er}$ is open in $\operatorname{Gr}(2, H^2(M, \mathbb{R}))$, **$G'$ is an open subgroup in the subgroup G'' of $SO^+(H^2(M, \mathbb{R}), q)$ generated by all rotations in $W \in \operatorname{Gr}(2, H^2(M, \mathbb{R}))$.**

The Lie group generated by Hodge rotations (2)

THEOREM: Let $G \subset \operatorname{Aut}(H^*(M,\mathbb{R}))$ be the subgroup generated by all Hodge rotations $\rho_I(t)$, for all complex structures I of hyperkähler type. **Then G acts on $H^2(M,\mathbb{R})$ as $SO^+(H^2(M,\mathbb{R}),q)$.**

Step 2: The group $G' := G\Big|_{H^2(M,\mathbb{R})}$ **is an open subgroup in the subgroup G'' of $SO^+(H^2(M,\mathbb{R}),q)$ generated by all rotations in $W \in \operatorname{Gr}(2, H^2(M,\mathbb{R}))$.**

Step 3: A map which is $O(H^2(M,\mathbb{R}),q)$-conjugated to a rotation in $W \in \operatorname{Gr}(2, H^2(M,\mathbb{R}))$ is again a rotation, hence G'' is a normal subgroup in $SO^+(H^2(M,\mathbb{R}),q)$. This group is simple for $\operatorname{rk} H^2(M,\mathbb{R}) \neq 4$, because the corresponding complex Lie algebra is simple. When $\operatorname{rk} H^2(M,\mathbb{R})$, the Lie algebra is $\mathfrak{so}(3)^2$, but the Lie group group $SO(1,3) = SL(2,\mathbb{C})/\pm 1$ is nevertheless simple. **We proved that $G'' = SO^+(H^2(M,\mathbb{R}),q)$.**

Step 4: This implies that G' is an open, connected subgroup of $SO^+(H^2(M,\mathbb{R}),q)$, but a connected Lie group is multiplicatively generated by any neighbourhood of the unity, **(prove this as an exercise)** hence $G' = SO^+(H^2(M,\mathbb{R}),q)$. ■

The Hodge-Riemann bilinear relations

DEFINITION: Let M be a compact Kähler manifold, and $p+q \leqslant n = \dim_{\mathbb{C}} M$. Denote by $H^*(M) = \bigoplus_{r=0}^n W_r$ the weight decomposition associated with the Lefschetz $\mathfrak{sl}(2)$-action, with W_r a direct sum of $\mathfrak{sl}(2)$-representations of weight r. Let $V_k^{p,q} := H^{p,q}(M) \cap W_k$. **Clearly,** $H^*(M) = \bigoplus_{k,p,q} V_k^{p,q}$**. The Riemann-Hodge form** on $V_k^{p,q}$ is

$$\eta, \eta' \longrightarrow \sqrt{-1}^{p-q}(-1)^{p+q-k} \int_M \eta \wedge \overline{\eta}' \wedge \omega^{n-p-q}$$

THEOREM: (Riemann-Hodge relations)
The Riemann-Hodge form is positive definite.

Proof: Follows from Weyl's structure theorem on tensor representations of $U(n)$. See *Howe, Roger E., "Remarks on classical invariant theory", Transactions of the American Mathematical Society, American Mathematical Society, 313 (2): 539-570, 1989.*

REMARK: For 2-forms **this statement is the Hodge index theorem,** proven earlier today.

The automorphisms of the cohomology algebra

THEOREM: The kernel of the natural restriction map $\operatorname{Aut}(H^*(M,\mathbb{R})) \longrightarrow O(H^2(M,\mathbb{R}),q)$ is a compact group.

Proof. Step 1: Let $u \in \operatorname{Aut}(H^*(M,\mathbb{R}))$ be an automorphism which trivially acts on $H^2(M,\mathbb{R})$. **Then u commutes with all Lefschetz triples.** Indeed, in an $\mathfrak{sl}(2)$-triple acting on a finite-dimensional vector space V, any two elements define the third **(prove this as an exercise)**, hence **any automorphism of $H^*(M,\mathbb{R})$ which preserves L and H also preserves Λ.**

Step 2: In Lecture 22, we proved that $W_I = [L_J, \Lambda_K]$, where W_I is the operator acting as $\sqrt{-1}\,(p-q)\operatorname{Id}$ on $H^{p,q}(M)$. Since u preserves the Lefschetz $\mathfrak{sl}(2)$, it also commutes with W_I, and hence **preserves the Hodge decomposition.**

Step 3: We obtain that u preserves the decomposition $H^*(M) = \bigoplus_{k,p,q} V_k^{p,q}$ and commutes with the Hodge-Riemann pairing; **the group of such automorphisms is compact, because the Hodge-Riemann pairing is positive definite.** ■

COROLLARY: The kernel of $\operatorname{Aut}(H^*(M,\mathbb{Z})) \longrightarrow O(H^2(M,\mathbb{Z}),q)$ is finite.

Proof: Indeed, $\operatorname{Aut}(H^*(M,\mathbb{Z}))$ is discrete, and a discrete subgroup of a compact group is finite. ■