K3 surfaces

lecture 2: Hopf theorem

Misha Verbitsky

IMPA, sala 236

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Graded vector spaces and algebras

DEFINITION: Graded vector space is a space $V^* = \bigoplus_{i\in\mathbb{Z}} V^i$.

REMARK: If V^* is graded, the endomorphism vector space $\operatorname{End}(V^*) = \bigoplus_{i\in\mathbb{Z}} \operatorname{End}^i(V^*)$ is also graded,

$$\operatorname{End}^i(V^*) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}(V^j, V^{i+j}).$$

DEFINITION: Graded algebra is an algebra $A^* = \bigoplus_{i\in\mathbb{Z}} A^i$ with multiplication which is compatible with the grading, $A^i \cdot A^j \subset A^{i+j}$.

REMARK: Bilinear map of graded vector spaces which satisfies $A^i \cdot B^j \subset C^{i+j}$, is called **compatible with the grading**.

REMARK: Category of graded vector spaces can be defined **is equivalent to the category of representations of $U(1)$**; the weight decomposition defines, and is defined, by an action $\rho(t)|_{A^n} = e^{2\pi\sqrt{-1}\,nt}$. Then **graded algebra is an associate algebra in the category of vector spaces with $U(1)$-action.**

Supercommutator

DEFINITION: An operator on a graded vector space is called **even** (**odd**), if it shifts the grading by an even (odd) number. **Parity** $\tilde{a}$ of an operator a is 0, if it is even, and 1 otherwise. We say that an operator is **pure** if it is even or odd.

DEFINITION: **Supercommutator** (or **graded commutator** of pure elements is defined by the formula $\{a,b\} = ab - (-1)^{\tilde{a}\tilde{b}}ba$.

DEFINITION: Graded associative algebra A^* is caled **supercommutative**, or **graded commutative**, if the supercommutator in A^* vanishes.

EXAMPLE: Grassmann algebra $\Lambda^* V$ is supercommutative.

Bialgebras

DEFINITION: A graded commutative, associative algebra A over a field k is called **a graded bialgebra** if A is equipped with a morphism of graded algebras $A \xrightarrow{\Delta} A \otimes_k A$ **("the compultiplication")**, which satisfies **the associativity condition:** $\Delta \otimes \operatorname{Id}_A \circ \Delta = \operatorname{Id}_A \otimes \Delta \circ \Delta : A \longrightarrow A \otimes_k A \otimes_k A$. **Counit** of a bialgebra is a k-algebra homomorphism $A \xrightarrow{\varepsilon} k$ which satisfies $\Delta \circ (\varepsilon \otimes \operatorname{Id}_A) = \Delta \circ (\operatorname{Id}_A \otimes \varepsilon) = \operatorname{Id}_A$.

REMARK: **Further on, all bialgebras are assumed to be with unit**.

REMARK: Coassoacitivity plus existence of a couniut means **that the dual space A^* is a graded algebra.** This coalgebra structure is compatible with multiplication in A means that **the algebra structure in A^* is a morphism of A-modules.**

Examples of bialgebras

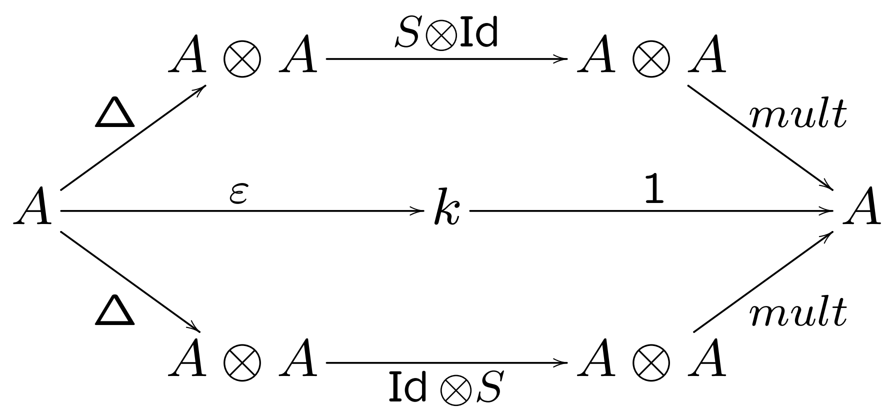
EXAMPLE: Let N be a set equipped with an associative, commutative operation $N \times N \xrightarrow{m} N$ (such a structure is called **the structure of a commutative, associative monoid**). Then **the ring of k-valued functions $C(N)$ is a bialgebra,** $m^* C(N) \longrightarrow C(N \times N) = C(N) \otimes_k C(N)$. Its is not graded.

REMARK: The notion of a bialgebra is an abstract version of this notion, heuristically, **bialgebras are function algebras on monoids**.

EXAMPLE: Let N be a connected topological space equipped with a continuous map $N \times N \xrightarrow{m} N$, defining on N a structure of commutative, associative monoid. Consider the comultiplication on its cohomology algebra $H^*(N)$, defined by the map $m^* : H^*(N) \longrightarrow H^*(N \times N) = H^*(N) \otimes_k H^*(N)$. **Then $H^*(N)$ is a graded bialgebra** (check this).

Hopf algebras

DEFINITION: A bialgebra is called **a Hopf algebra** if it is equipped with a homomorphism $A \xrightarrow{S} A$ **("the antipode map")**, and the following diagram is commutative:



REMARK: The antipode condition is self-dual: if A is a Hopf algebra, the dual space A^* is also a Hopf algebra, multiplication goes to comultiplication.

EXAMPLE: Let N be a group, and $C(N)$ the space of functions on N equipped with the bialgebra structure Then the map $n \longrightarrow n^{-1}$ defines an antipode structure on $C(N)$. We obtain that **the algebra $C(N)$ of functions on a group is a Hopf algebra** (check this).

EXAMPLE: Let G be a topological group, and $H^*(G)$ its cohomology algebra. Consider the map $H^*(G) \xrightarrow{S} H^*(G)$, induced by $x \longrightarrow x^{-1}$. **Then $H^*(G)$ is a Hopf algebra** (check this).

H-groups

DEFINITION: An H-group is a topological space M, equipped with a map $M\times M \xrightarrow{\mu} M$, **("multiplication map")**, and a point $e\in M$ **("the homotopy unit")** such that the restriction of μ to $M\times\{e\}$ is homotopy equivalent to identity, and and a map $M \xrightarrow{\eta} M$ **("homotopy inverse map")**, which satisfies the group axiom **up to homotopy**:

* **Homotopy associativity:** a the maps $\mu\times \operatorname{Id}\circ\mu$ and $\operatorname{Id}\times\mu\circ\mu$ are homotopy equivalent as maps $M\times M\times M\mapsto M$,

* **the homotopy inverse**: the compositions of $\operatorname{diag}\circ(\eta\times\operatorname{Id})\circ\mu$ and $\operatorname{diag}\circ(\operatorname{Id}\times\eta)\circ\mu$ are homotopy to a map $M\mapsto pt$ to a point.

EXAMPLE: Check that **the loop space $\Omega(X,x)$ is an H-group.**

CLAIM: Let M be an H-group. Then the cohomology algebra $H^*(M,k)$ **is a graded Hopf algebra.**

Structure theorem for Hopf algebras

Let $V^\bullet$ be a graded vector space, and $\operatorname{Sym}_{gr}(V)$ the tensor product $\operatorname{Sym}^*(V^{\text{even}})\otimes \Lambda^*(V^{\text{odd}})$ with a natural grading. The space $\operatorname{Sym}_{gr}(V)$ is equipped with a structure of graded commutatiove algebra.

DEFINITION: Free commutative algebra generated by V is $\operatorname{Sym}^*(V)$ (the polynomial algebra). **Free graded commutative algebra** is $Sym_{gr}(V^\bullet)$, where $V^\bullet$ is a graded vector space.

DEFINITION: Graded algebra of finite type is algebra, graded by $i \geqslant 0$, with all graded components finite-dimensional.

Hopf theorem: Let A be a graded Hopf algebra of finite type over a field k of characteristic 0. **Then A is a free graded commutative algebra.**

Primitive elements in bialgebras

DEFINITION: An element x of a bialgebra is called **primitive**, if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

First, we prove Hopf theorem for Hopf algebras gnerated by primitives.

DEFINITION: Let A be a Hopf algebra, and $P \subset A$ the space of primitive elements. Consider the natural multiplicative homomorphism $\operatorname{Sym}_{gr}(P) \xrightarrow{\psi} A$. We say that A **is free in degrees up to k**, if $\bigoplus^{i\leqslant k} \operatorname{Sym}^i_{gr}(P) \xrightarrow{\psi} A$ is injective

REMARK: The following lemma **immediately implies Hopf theorem for any algebra generated by primitives.**

LEMMA 1: Let A be a Hopf algebra which is free up to degree k. k. **Then A is free up to degree $k+1$.**

Structure theorem for Hopf algebras generated by primitives

LEMMA 1: Let A be a Hopf algebra which is free up to degree k. k. **Then A is free up to degree $k+1$.**

Proof. Step 1: Let $\{x_i\}$ be a basis in the space P of primitive elements. Consider a polynomial relation of degree $k+1$. $Q(x_1, ..., x_n) = 0$. We write Q as a polynomial in x_1 with coefficients in $x_2, ..., x_n$: $Q = Q_m x_1^m + Q_{m-1} x_1^{m-1} + ... + Q_0$. Since $\psi : \bigoplus^{i \leqslant k} \operatorname{Sym}^i_{gr}(P) \stackrel{\psi}{\longrightarrow} A$ is injective, we have

$$\Delta(Q) = Q \otimes 1 + 1 \otimes Q + R,$$

where $R \in \mathfrak{A} := \left(\bigoplus^{i \leqslant k} \operatorname{Sym}^i_{gr}(P)\right) \otimes \left(\bigoplus^{i \leqslant k} \operatorname{Sym}^i_{gr}(P)\right)$. Note that the natural map $\mathfrak{A} \to A \otimes A$ is injective, because A is free in degrees $\leqslant k$.

Step 2: Every element of $\mathfrak{A}$ can be represented as a sum of monomials $\lambda \otimes \mu$, where λ, μ are monomials in x_i. Ley $\Pi : \mathfrak{A} \longrightarrow x_1 \otimes \left(\bigoplus^{i \leqslant k} \operatorname{Sym}^i_{gr}(P)\right)$ be the projection to the sum of all monomials $x_1 \otimes \mu$. Since $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$, we have $\Delta(x_1^m) = (x_1 \otimes 1 + 1 \otimes x_1)^m$, **giving $\Pi(\Delta(x_1^m)) = m x_1 \otimes x_1^{m-1}$.**

Structure theorem for Hopf algebras generated by primitives (2)

Step 3: Let $\Pi(R) = x_1 \otimes R_0$. Since $Q = 0$ in A, its component R_0 vanishes. **By Step 2,**

$$0 = x_1 \otimes R_0 = \sum_{i=1}^{m} m x_1^{m-1} Q_m$$

where Q_i are polynomials defined in Step 1. Therefore, all $Q_i = 0$. ■

REMARK: In step 3 **we use** $\operatorname{char} k = 0$**.**

Proof of Hopf theorem

DEFINITION: Augmentation ideal Z in Hopf algebra is the kernel of the counit map $\varepsilon : A \longrightarrow k$.

REMARK: The counit condition gives $x = [\varepsilon \otimes \mathrm{Id}_A](\Delta(x))$ and $x = [\mathrm{Id}_A \otimes \varepsilon](\Delta(x))$, hence

$$\Delta(x) = 1 \otimes x + x \otimes 1 \mod (Z \otimes Z)$$

Proof of Hopf theorem. Step 1: Consider a filtration of A by the powers Z^i of the augmentation ideal Z, and let $A_{gr} := \bigoplus_i Z^i / Z^{i+1}$ be the associated graded algebra. **By the remark above, we have $\Delta(Z^p) \subset \bigoplus_{i+j=p} Z^i \otimes Z^j$.**

Step 2: This implies that all operations, used in the definition of Hopf algebra, are compatiblle with the filtration defined by the powers of Z (check this). Therefore, A_{gr} **is also a Hopf algebra**.

Step 3: The algebra A_{gr} is multiplicatively generated by Z^1/Z^2 (this is a general properties of algebras filtered by degrees of an ideal).

Proof of Hopf theorem (2)

Step 4: Since $\Delta(x) = 1 \otimes x + x \otimes 1 \mod (Z \otimes Z)$, all elements of Z^1/Z^2 are primitive in A_{gr}. By Step 3, A_{gr} **is generated by primitive elements.**

Step 5: By Lemma 1, A_{gr} is freely generated by $\{x_i\}$, where x_i is a basis in the space of primitive elements in A_{gr}. Choose for each x_i a representative $\tilde{x}_i$ in A of the same parity. Since there are no non-trivial relations between x_i inA_{gr}, **there are no non-trivial relations between $\tilde{x}_i$ in A. It remains to show that $\tilde{x}_i$ generate A.**

Step 6: The dimensions of the graded spaces A^p and A^p_{gr} are equal **here we speak of the grading which was initially defined on A, and the grading on A_{gr} induced from the grading on A**). Let $\{y_i\}$ be a collection of monomials in the free graded algebra A_{gr}, generating a given component A^p_{gr}, and $\{\tilde{y}_i\}$ – the corresponding elements in A^p. Then $\{y_i\}$ is a basis A^p_{gr}, and $\{\tilde{y}_i\}$ linearly independent elements of the space A^p of the same dimension. Therefore, **the monomials $\{\tilde{y}_i\}$ generate A^p.** ■

Applications of Hopf theorem

COROLLARY: The algebra $H^*(G, \mathbb{Q})$ of cohomology of a Lie group **is isomorphic to a Grassmann algebra.**

COROLLARY: The algebra $H^*(\Omega M, \mathbb{Q})$ of cohomology of the space of loops of a finite cell space M **is a free supercommutative algebra.**

REMARK: The proof of the structure theorem **never uses the antipode axiom.** Therefore, **Hopf theorem is true for any bialgebra of finite type**, in particular, **for the cohomology algebra of any space with homotopy associative multiplication and a homotopy unit.**

Cohomology algebra of $U(n)$

CLAIM: The cohomology algebra of $H^*(U(n),\mathbb{Q})$ – **is a free graded commutative algebra with generators in degrees $1,3,5,...,2n-1$.**

Proof. Step 1: Since $U(n)$ is a Lie group, its cohomology is a Hopf algebra. Therefore, $H^*(U(n-1))$ **is a free graded commutative algebra with generators in odd degrees.** Using induction, we can assume that $H^*(U(n-1))$ **– is a free algebra with generators in degrees $1,3,5,...,2n-3$.**

Step 2: The group $U(n)$ is fibered over S^{2n-1} with fiber $U(n-1)$, and the form $\xi_{2n-1} \in H^*(U(n))$, generated by a pullback of the volume form on a sphere is closed. Since ξ_{2n-1} vanishes on vectors tangent to $U(n-1)$, there are no relations between ξ_{2n-1} and elements of $H^*(U(n-1))$. Therefore, $H^*(U(n))$ contains a free algebra A^*, generated by $H^*(U(n-1))$ and ξ_{2n-1}. Writing a cell decomposition associated with the fibration $U(n) \stackrel{U(n-1)}{\longrightarrow} S^{2n-1}$, we show that the dimension of A^* is $\dim H^*(U(n))$, which gives $A^* = H^*(U(n))$. ■