K3 surfaces

lecture 3: topology of complex surfaces

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Topology of 4-manifolds

REMARK: Topologically, there are 3 interesting classes of manifolds: **smooth manifolds**, **PL manifolds** (that is, manifolds with triangulation) and **smooth manifolds**. Any two diffeomorphic smooth manifolds are PL-equivalent, but PL-manifolds might have non-equivalent smooth structures. There are PL-manifolds not admitting smooth structures as well. Any topological manifold might have non-equivalent PL-structures; there are also topological manifolds admitting no PL (and hence, no smooth) structures. This hierarchy is well understood outside of dimension 4. In dimension 4 it is not well understood, but we know that **even $\mathbb{R}^4$ has uncountable number of pairwise non-equivalent smooth structures (Donaldson)**.

Topological 4-manifolds are well understood, due to M. Freedman (Freedman got the Fields medal for this work). Then Donaldson has shown that the topology of **smooth** manifolds **is much more complicated** (and still mysterious). He also got Fields medal for this. The idea is that the classification of smooth structures **is closer to algebraic and symplectic geometry** than to the topology.

I will explain these results **as far as they can be applied to K3 surfaces.**

The intersection form

DEFINITION: Let V be $\mathbb{Z}^n$, considered as a $\mathbb{Z}$-module. A bilinear symmetric form $\eta : V \otimes_{\mathbb{Z}} V \longrightarrow \mathbb{Z}$ is called **unimodular** if it defines an isomorphism $V \to \operatorname{Hom}_{\mathbb{Z}}(V, \mathbb{Z})$, **even** if $\eta(x,x)$ is even for all $x \in V$, and **odd** otherwise. **Signature** of η is signature of the corresponding bilinear form on the vector space $R^n = V \otimes_{\mathbb{Z}} \mathbb{R}$.

THEOREM: (Universal coefficients formula)
Let X be a cellular topological space, $b_i(X)$ its Betti numbers, and T_i the torsion subgroup in $H_i(X,\mathbb{Z})$. **Then $H^n(X,Z) = \mathbb{Z}^{b_n(X)} \oplus T_{n-1}(X)$.**

COROLLARY: Let X be a simply connected cellular space (a manifold or a variety). **Then $H^2(X,\mathbb{Z})$ is torsion-free.**

DEFINITION: Let M be a 4-manifold. **Intersection form** of M is the form $x, y \mapsto \int_M x \cap y$ on the torsion-free part of $H^2(M,\mathbb{Z})$. **Signature** of M is the signature of this bilinear symmetric form.

REMARK: By Poincaré duality, **the intersection form is unimodular.**

THEOREM: (Rokhlin, Wu)
Let M be a smooth, simply connected 4-manifold with even intersection form. **Then its signature is divisible by 16.**

Topology of 4-manifolds and Freedman's theorem

THEOREM: (M. Freedman, 1982) The homotopy class of a compact, simply connected 4-manifold M **is uniquely determined by its intersection form** $H^2(M,\mathbb{Z}) \otimes_{\mathbb{Z}} H^2(M,\mathbb{Z}) \longrightarrow \mathbb{Z}$**.** Moreover, **such** M **exists for any unimodular form**. For even intersection forms, homotopy equivalence is equivalent to homeomorphism. For odd intersection forms, there exists exactly two topological manifolds with a given homotopy type; one of them admits a PL-structure, another does not.

For smooth manifolds, **the situation is entirely different.**

THEOREM: (Donaldson, 1986) Let M be a smooth compact manifold with odd, positive definite intersection form η. **Then** η **admits a diagonalization,** that is, for some integer basis $x_i \in H^2(M,\mathbb{Z})^*$, one has $\eta = \sum x_i \otimes x_i$.

Odd and even unimodular quadratic forms

REMARK: Generally speaking, a positive definite unimodular form, even or odd, **cannot be diagonalized,** hence Donaldson's theorem **is very restrictive.**

EXAMPLE: Clearly, **an even form cannot be diagonalized.** This includes, for example, the form $\eta_{\mathbb{Z}/2} := \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$, **which is clearly unimodular,** because its determinant is -1.

REMARK: The number of isomorphism classes of positive definite unimodular lattices (odd and even) **grows very fast.** **The sequence A054911 from OEIS** (number of n-dimensional odd unimodular lattices): 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 3, 3, 4, 5, 6, 9, 13, 16, 28, 40, 68, 117, 273, 665, 2566, 17059, 374062. **The sequence A054909 from OEIS:** (number of $8n$-dimensional odd unimodular lattices): 1, 2, 24, $\geqslant$ 1162109024.

Classification of indefinite bilinear symmetric forms.

DEFINITION: A bilinear symmetric form η is called **indefinite**, if $\eta(x,x)<0$ and $\eta(y,y)>0$ for some x and y.

THEOREM: (classification of unimodular bilinear symmetric forms)
* Let q be an odd unimodular indefinite form. Then q is diagonal: $q = \sum \pm x_i \otimes x_i$.
* Let q be an even unimodular indefinite form. **Then (V,q) can be represented as a direct sum** of **quadratic lattices** (that is, $\mathbb{Z}$-modules with bilinear forms) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and **quadratic lattices $E_{\pm 8}$,**. The bilinear form E_8 is isomorphic to the intersection form of the Cartan algebra of the special Lie group E_8:

$$\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 2
\end{pmatrix},$$

and $-E_8$ is the same form with the opposite sign.

4-dimensional manifolds as connected sums

DEFINITION: Connected sum $M_1\#M_2$ of manifolds M_1 and M_2 of the same dimension is constructed as follows. We remove a ball from M_1 and from M_2, and glue the corresponding spherical boundary components of these manifolds.

CLAIM: Let M_1, M_2 be manifolds with the intersection form q_1, q_2. Then for all $0 < i < \dim M$ we have $H^i(M_1\#M_2) = H^i(M_1) \oplus H^i(M_2)$, and **the intersection form on $M_1\#M_2$ is equal to $q_1 \oplus q_2$.**

REMARK: Donaldson and Freedman's theorem imply that **any smooth 4-manifold with odd intersection form is homeomorphic to a connected sum of several copies of $\mathbb{C}P^2$ (the intersection form on $\mathbb{C}P^2$ is diagonal with signature 1), and several copies of the manifold $\overline{\mathbb{C}P^2}$,** obtained from $\mathbb{C}P^2$ by the change of orientation (the intersection form is diagonal with signature -1).

4-dimensional manifolds with even intersection form

DEFINITION: **E_8-manifold** is a simply connected 4-manifold with the intersection form E_8. By Rokhlin's theorem, **E_8-manifold does not admit a smooth structure** (its signature is not divisible by 16). The E_8-manifold with opposite orientation is called **the $-E_8$-manifold**.

REMARK: From the classification of even unimodular forms and Freedman's theorem it follows immediately that **every 4-manifold with even, indefinite intersection form is homeomorphic to a connected sum of several copies of E_8-manifolds, $-E_8$-manifolds, and $\S^2 \times S^2$.**

DEFINITION: **A topological 4-manifold if K3 type** can be defined as a connected sum of two $-E_8$-manifolds and three $S^2 \times S^2$. **It admits a smooth structure (an infinite countable number of smooth structures, in fact).**

REMARK: It is still not entirely clear **when a manifold of form $\pm E_8^k \# (S^2 \times S^2)^l$ admits a smooth structure.**

THEOREM: (Furuta) If $E_8^{2k} \# (S^2 \times S^2)^l$ admits a smooth structure, then $l > 2k$.

Complex manifolds (reminder)

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

REMARK: The commutator defines a $\mathbb{C}^\infty M$-linear map $N := \Lambda^2(T^{1,0}) \longrightarrow T^{0,1}M$, called **the Nijenhuis tensor** of I. **One can represent N as a section of $\Lambda^{2,0}(M) \otimes T^{0,1}M$.**

Exercise: Prove that **$\mathbb{C}P^n$ is a complex manifold,** in the sense of the above definition.

Kähler manifolds (reminder)

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

REMARK: It is $U(1)$-invariant, hence **of Hodge type (1,1)**.

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

Calabi-Yau manifolds

REMARK: Let B be a line bundle on a manifold. Using the long exact sequence of cohomology associated with the exponential sequence

$$0 \longrightarrow \mathbb{Z}_M \longrightarrow C^\infty M \longrightarrow (C^\infty M)^* \longrightarrow 0,$$

we obtain a short exact sequence $0 \to H^1(M, (C^\infty M)^*) \xrightarrow{\xi} H^2(M, \mathbb{Z}) \to 0$**,** hence every bundle B **is uniquely determined by the cohomology class** $\xi_B \in H^2(M, \mathbb{Z})$**.**

DEFINITION: Let B be a complex line bundle, and ξ_B its defining element in $H^1(M, (C^\infty M)^*) = H^2(M, \mathbb{Z})$. Its image in $H^2(M, \mathbb{Z})$ is called **the first Chern class** of B.

REMARK: A complex line bundle B is topologically trivial $\Leftrightarrow c_1(B) = 0$.

DEFINITION: The first Chern class of a complex n-manifold is $c_1(\Lambda^{n,0}(M))$.

DEFINITION:
A Calabi-Yau manifold is a compact Kaehler manifold with $c_1(M, \mathbb{Z}) = 0$.

K3 surfaces

DEFINITION: A complex surface is a compact, complex manifold of complex dimension 2.

DEFINITION: A K3 surface is a Kähler complex surface M with $b_1 = 0$ and $c_1(M, \mathbb{Z}) = 0$.

REMARK: All surfaces with b_1 even are Kähler (Kodaira, Buchdahl-Lamari).

The name K3 is given by Andre Weil in honor of Kummer, Kähler and Kodaira.

The Broad Peak

"Faichan Kangri (K3) is the 12th highest mountain on Earth."

Chez les Weil. André et Simone

André Weil: 6 May 1906 - 6 August 1998.

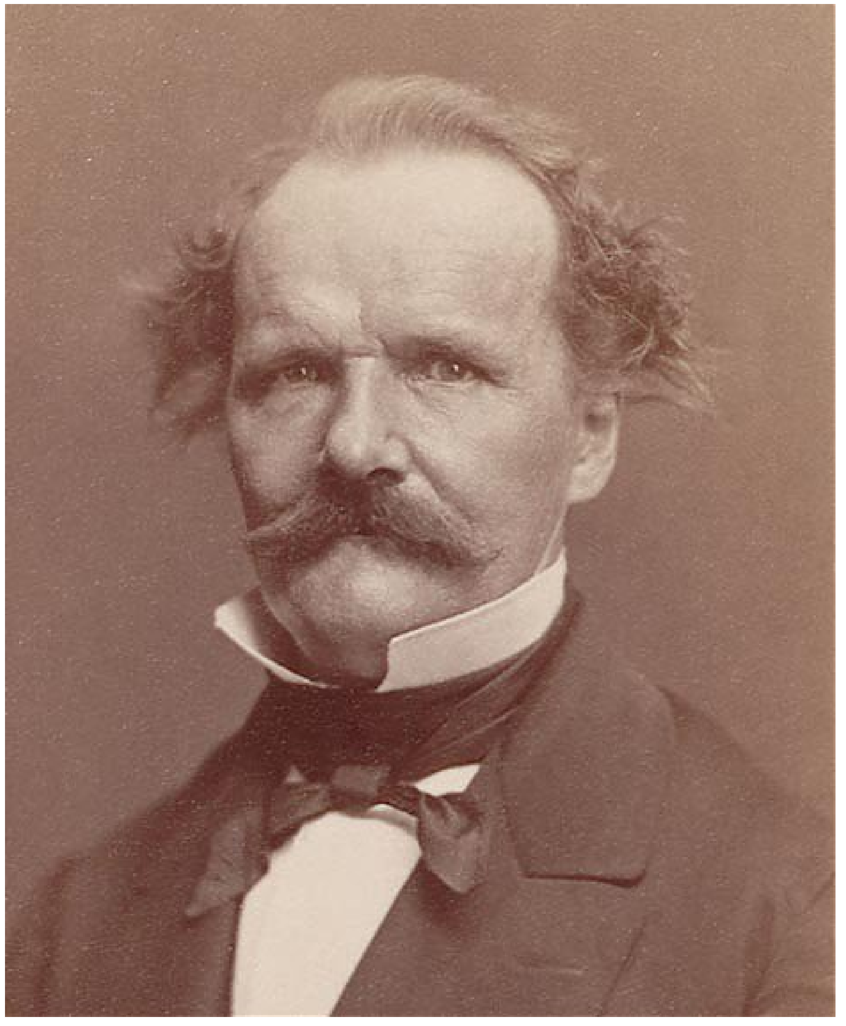
"Simone et André à Penthiévre, 1918-1919"

Erich Kähler

(Erich Kähler: 1990)

16 January 1906 - 31 May 2000

Ernst Eduard Kummer



Ernst Kummer: 29 January 1810 - 14 May 1893

Kunihiko Kodaira

(Kunihiko and Seiko Kodaira)

Kunihiko Kodaira: 16 March 1915 - 26 July 1997

Properties of K3 surfaces

CLAIM: Let M be a K3 surface, and $K_M := \Lambda^2(\Omega^1 M)$ be its canonical bundle. **Then K_M is trivial as a holomorphic vector bundle.**

Proof: Since $b_1 = 0$ and M is Kähler, we have $h^{0,1} = H^1(\mathcal{O}_M) = 0$. Then the exponential exact sequence $H^1(M, \mathcal{O}_M) \longrightarrow H^1(M, \mathcal{O}_M^*) \xrightarrow{c_1} H^2(M, \mathbb{Z})$ implies that K_M **is trivial** (its Chern class vanishes). ■

REMARK: Let $\chi(M) = \sum_i (-1)^i \dim H^i(M, \mathcal{O}_M)$ be **the holomorphic Euler characteristic.** Riemann-Roch formula for surfaces gives $\chi(M) = \frac{c_1^2 + c_2}{12}$. Applying this to K3 and using $H^1(\mathcal{O}_M) = 0$ and $H^2(\mathcal{O}_M) = H^0(K_M)^* = \mathbb{C}$ (Serre's duality), we obtain that $\chi(\mathcal{O}_M) = 2$. Since $c_1(M) = 0$, this implies $\chi(M) = 2 = \frac{c_2(M)}{12}$, giving $c_2(M) = 24$. Since $c_2(M)$ is the Euler characteristic of M, we obtain $b_2(M) = 22$.

This gives the Hodge diamond for a K3 surface:

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$