K3 surfaces

lecture 4: Teichmüller spaces

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Geometric structures

DEFINITION: "Geometric structure" on a manifold M is a reduction of its structure group $GL(n,\mathbb{R})$ to a subgroup $G \subset GL(n,\mathbb{R})$. However, it is easier to define it by a collection of tensors $\Psi_1, ..., \Psi_n$ such that the stabilizer $\operatorname{St}_{\langle\Psi_1,...,\Psi_n\rangle} \subset GL(T_xM)$ of $\Psi_1, ..., \Psi_n$ at each point $x \in M$ **is conjugate to the same group $G \subset GL(n,\mathbb{R})$.** Usually, in addition to this algebraic condition, people ask for some differential conditions to hold, such as the integrability for almost complex structures.

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $[T^{0,1}M, T^{0,1}M] \subset T^{0,1}M$.

DEFINITION: Symplectic form on a manifold is a non-degenerate differential 2-form ω satisfying $d\omega = 0$.

Today I would define **the Techmüller space** of geometric structures and describe it for some examples.

Fréchet spaces

DEFINITION: A seminorm on a vector space V is a function $\nu: V \longrightarrow \mathbb{R}^{\geqslant 0}$ satisfying

1. $\nu(\lambda x) = |\lambda|\nu(x)$ for each $\lambda \in \mathbb{R}$ and all $x \in V$

2. $\nu(x+y) \leqslant \nu(x) + \nu(y)$.

DEFINITION: We say that **topology on a vector space V is defined by a family of seminorms $\{\nu_\alpha\}$** if the base of this topology is given by the finite intersections of the sets

$$B_{\nu_\alpha,\varepsilon}(x) := \{y \in V \quad | \quad \nu_\alpha(x-y) < \varepsilon\}$$

("open balls with respect to the seminorm"). It is **complete** if each sequence $x_i \in V$ which is Cauchy with respect to each of the seminorms converges.

CLAIM: A topology on V defined by a family of seminorms $\{\nu_\alpha\}$ **is Hausdorff if and only if for each $v \in V$ there exists a seminorm $\nu \in \{\nu_\alpha\}$ such that $\nu(v) \neq 0$.** ■

Fréchet spaces and translation-invariant metrics

DEFINITION: A **Fréchet space** is a Hausdorff second countable topological vector space V with the topology which can be defined by a countable family of seminorms, complete with respect to this family of seminorms.

DEFINITION: Equivalent definition: let V be a vector space equipped with a collection of norms (or seminorms) $|\cdot|_i$, $i = 0, 1, 2, ...$ and a topology which is given by the metric $d(x, y) = \sum_{i=0}^{\infty} 2^{-i} \min(|x - y|_i, 1)$, assumed to be non-degenerate. The space V is called **a Fréchet space** if this metric is complete.

REMARK: Completeness **is equivalent to convergence of any sequence $\{a_i\}$ which is fundamental with respect to all the (semi-)norms $|\cdot|_i$.**

REMARK: A sequence converges in the Fréchet topology given by d $\Leftrightarrow$ it converges in any of the (semi-)norms $|\cdot|_i$.

EXERCISE: Let V be a vector space, equipped with a translation-invariant metric d. Assume that the open balls are convex, and V is complete and second countable with respect to d. **Prove that V is Fréchet, and all Fréchet spaces can be obtained this way.**

C^∞-topology

DEFINITION: Let M be a Riemannian manifold, and $\nabla^i: C^\infty(M) \longrightarrow \Lambda^1(M)^{\otimes i}$ the iterated connection. **Topology C^k** on the space $C^\infty_c(M)$ of functions with compact support is defined by the norm

$$|\varphi|_{C^k} := \sup_M \sum_{i=0}^k |\nabla^i \varphi|.$$

EXERCISE: Prove that the space $C^\infty_c M$ of functions with compact support **is a Fréchet space with respect to C^∞-topology.**

REMARK: This topology **is independent from the choice of the connection.** This is an exercise.

REMARK: A tensor on a manifold is a section of the tensor bundle $TM^{\otimes i} \otimes T^*M^{\otimes j}$. The same way one defines **the C^∞-topology on the space of tensors with compact support on M.**

EXERCISE: Prove that **the space of tensors with compact support is a Fréchet space,** with the C^∞-topology defined as above.

C^0-topology on the group of diffeomorphisms

DEFINITION: Let M be a compact Riemannian manifold. **The C^0-topology** on the space of diffeomorphisms is defined by the metric $d(\tau_1, \tau_2) := \sup_{x\in M} d(\tau_1(x), \tau_2(x))$.

EXERCISE: Prove that **this topology is independent from the choice of Riemannian structure.**

EXERCISE: Prove that **the group of homeomorphisms is complete with respect to d.**

REMARK: This topology is not enough for many purposes, for example, **the map $\tau \longrightarrow D_x\tau$ is not continuous in C^0-topology,** because it depends on the derivative of the diffeomorphism.

C^∞-topology on the group of diffeomorphisms

We define **C^∞-topology** on diffeomorphisms; it is **strictly stronger** (has more open sets) than the C^0-topology. We define it in such a way that the group structure on $\operatorname{Diff}(M)$ is compatible with the C^∞-topology. Then **it would suffice to define topology on a sufficiently small C^0-neighbourhood of $\operatorname{Id} \in \operatorname{Diff}(M)$.**

DEFINITION: Choose two atlases $\{U_i\}$ and $\{V_i\}$ on M, with **the closure of each U_i compact in V_i.** Then **there exists a C^0-neighbourhood $\mathcal{U}$ of $\operatorname{Id} \in \operatorname{Diff}(M)$ such that for all $\tau \in \mathcal{U}$ we have $\tau(U_i) \subset V_i$.** We define the C^∞-topology on $\mathcal{U}$ and expand it to $\operatorname{Diff}(M)$ using the group structure. For each $\tau \in \mathcal{U}$, we can interpret τ as a map from U_i to V_i, that is, as a collection of smooth functions. **The C^∞-topology on U is defined by uniform convergence of these functions with all their derivatives,** that is, by C^∞-topology on $\prod_i C^\infty(U_i, V_i)$.

THEOREM: **The C^∞-topology on diffeomorphisms is independent from the choices we made. The diffeomorphism group with respect to this topology is a Fréchet-Lie group.**

Proof: Left as an exercise.

Teichmüller space of geometric structures

Let $\mathcal{C}$ be the set of all geometric structures of a given type, say, complex, or symplectic. We put C^∞-topology (the topology of uniform convergence with all derivatives) on $\mathcal{C}$. Let $\operatorname{Diff}_0(M)$ be the connected component of its diffeomorphism group $\operatorname{Diff}(M)$ (**the group of isotopies**).

DEFINITION: The quotient $\mathcal{C}/\operatorname{Diff}_0$ is called **Teichmüller space** of geometric strictures of this type.

DEFINITION: The group $\Gamma := \operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ is called **the mapping class group** of M. It acts on Teich by homeomorphisms.

DEFINITION: The orbit space $\mathcal{C}/\operatorname{Diff} = \operatorname{Teich}/\Gamma$ is called **the moduli space** of geometric structure of this type.

Today I will describe Teich in some interesting cases.

Teichmüller space for symplectic structures

DEFINITION: Let $\Gamma(\Lambda^2 M)$ be the space of all 2-forms on a manifold M, and $\operatorname{Symp} \subset \Gamma(\Lambda^2 M)$ the space of all symplectic 2-forms. We equip $\Gamma(\Lambda^2 M)$ with C^∞-topology of uniform convergence on compacts with all derivatives. Then $\Gamma(\Lambda^2 M)$ is a Frechet vector space, and Symp a Frechet manifold.

DEFINITION: Consider the group of diffeomorphisms, denoted Diff or $\operatorname{Diff}(M)$, as a Frechet Lie group, and denote its connected component ("group of isotopies") by Diff_0. The quotient group $\Gamma := \operatorname{Diff} / \operatorname{Diff}_0$ is called **the mapping class group** of M.

DEFINITION: **Teichmüller space of symplectic structures on M** is defined as a quotient $\operatorname{Teich}_s := \operatorname{Symp} / \operatorname{Diff}_0$. The quotient $\operatorname{Teich}_s / \Gamma = \operatorname{Symp} / \operatorname{Diff}$, is called **the moduli space of symplectic structures.**

REMARK: In many cases Γ acts on Teich_s with dense orbits, hence **the moduli space is not always well defined.**

DEFINITION: Two symplectic structures are called **isotopic** if they lie in the same orbit of Diff_0.

Moser's theorem

DEFINITION: Define **the period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ mapping a symplectic structure to its cohomology class.

THEOREM: (Moser, 1965)
The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ **is locally a diffeomorphism.**

The proof is based on another theorem of Moser.

Theorem 1: (Moser)
Let ω_t, $t \in S$ be a smooth family of symplectic structures, parametrized by a connected manifold S. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then all ω_t are isotopic.**

Proof of Moser theorem: The period map $P: \operatorname{Symp} \longrightarrow H^2(M,\mathbb{R})$ is a smooth submersion. By Theorem 1, the conneced components of the fibers of P are orbits of $\operatorname{Diff}_0(M)$. Therefore, Per is locally a diffeomorphism. ■

Symplectic structures on a compact torus

DEFINITION: A symplectic structure ω on a torus is called **standard** if there exists a flat torsion-free connection preserving ω.

REMARK: Moser's theorem immediately implies that **the set** $\operatorname{Teich}_{st}$ **of standard symplectic structures is open in the Teichmüller space.** Indeed, the period map from $\operatorname{Teich}_{st}$ to $H^2(M)$ is also locally a diffeomorphism.

REMARK: It is not known if any non-standard symplectic structures exist (even in dimension =4).

THEOREM: Let $\Lambda^2_{nd}(H_1(T)) \subset H^2(T)$ be the space of symplectic forms on $H_1(T)$, where T is an even-dimensional torus. Consider the period map $\operatorname{Per}: \operatorname{Teich}_{st} \longrightarrow \Lambda^2_{nd}(H_1(T)) \subset H^2(T)$, where $\operatorname{Teich}_{st}$ is the Teichmüller space of standard symplectic structures on T. **Then** Per **is a diffeomorphism on each connected component of** $\operatorname{Teich}_{st}$**.**

Proof: Left as an exercise.

The kernel of a differential form

DEFINITION: Let Ω be a differential form on M. The **kernel**, or **the null-space** $\ker(\Omega) \subset TM$ of Ω is the space of all vector fields $X \in TM$ such that the contraction $i_X(\Omega)$ vanishes.

Proposition 1: Let Ω be a closed form on a manifold, and $B \subset TM$ its null-space. **Then $[B, B] \subset B$.**

Proof. Step 1: Let $X, X_1 \in \ker(\Omega)$, and $X_2, ..., X_p$ any vector fields. Cartan's formula implies that $\operatorname{Lie}_X(\Omega) = d(i_X(\Omega)) + i_X(d\Omega) = 0$, hence $\operatorname{Lie}_X(\Omega) = 0$.

Step 2: $\operatorname{Lie}_X(\Omega)(X_1, ..., X_p) = \operatorname{Lie}_X(\Omega(X_1, ..., X_p)) - \sum_{i=1}^p \Omega(X_1, ..., [X, X_i], ...X_p)$. All terms of this sum, except $\Omega([X, X_1], X_2, ..., X_p)$, vanish, because $X_1 \in \ker(\Omega)$. Since $\operatorname{Lie}_X(\Omega) = 0$, we have $\Omega([X, X_1], X_2, ..., X_p) = 0$ for all $X_2, ..., X_p$. Therefore, $[X, X_1] \in \ker(\Omega)$. ■

Holomorphic symplectic form

DEFINITION: Holomorphic symplectic form on an almost complex manifold (M,I) is a non-degenerate closed differential 2-form $\Omega\in\Lambda^2(M,\mathbb{C})$ satisfying $d\Omega=0$ and $\Omega(Ix,y)=\sqrt{-1}\,\Omega(x,y)$.

REMARK: Consider the Hodge decomposition $T_{\mathbb{C}}M=T^{1,0}M\oplus T^{0,1}M$ (decomposition according to eigenvalues of I). Since $\Omega(IX,Y)=\sqrt{-1}\,\Omega(X,Y)$ and $I(Z)=-\sqrt{-1}\,Z$ for any $Z\in T^{0,1}(M)$, we have $\ker(\Omega)\supset T^{0,1}(M)$. Since $\ker\Omega\cap T_{\mathbb{R}}M=0$, real dimension of its kernel is at most $\dim_{\mathbb{R}}M$, giving $\dim_{\mathbb{R}}\ker\Omega=\dim M$. **Therefore,** $\ker(\Omega)=T^{0,1}M$**.**

COROLLARY: Let (M,I) be an almost complex manifold admitting a holomorphic symplectic form. **Then I is integrable** (Proposition 1.)

COROLLARY: Let Ω be a holomorphically symplectic form on a complex manifold (M,I). **Then I is determined by Ω uniquely.**

C-symplectic structures

DEFINITION: Let M be a smooth $4n$-dimensional manifold. A closed complex-valued form Ω on M is called **C-symplectic** if $\Omega^{n+1}=0$ and $\Omega^n\wedge\overline{\Omega}^n$ is a non-degenerate volume form.

THEOREM: Let $\Omega\in\Lambda^2(M,\mathbb{C})$ be a C-symplectic form, and $T^{0,1}_\Omega(M)$ be equal to $\ker\Omega$, where

$$\ker\Omega := \{v\in TM\otimes\mathbb{C} \ \ |\ \ \Omega\lrcorner v=0\}.$$

Then $T^{0,1}_\Omega(M)\oplus\overline{T^{0,1}_\Omega(M)}=TM\otimes_{\mathbb{R}}\mathbb{C}$, hence **the sub-bundle $T^{0,1}_\Omega(M)$ defines an almost complex structure I_Ω on M.** If, in addition, **Ω is closed, I_Ω is integrable, and Ω is holomorphically symplectic on (M,I_Ω).**

Proof: Rank of Ω is $2n$ because $\Omega^{n+1}=0$ and $\Omega^n\wedge\overline{\Omega}^n$ is non-degenerate. Therefore, $\operatorname{rk}T^{0,1}_\Omega(M)=2n$. For any $v\in T^{0,1}_\Omega(M)\cap\overline{T^{0,1}_\Omega(M)}$, the real part $\operatorname{Re}v$ of v belongs to $\ker\Omega$. This would imply that $\operatorname{Re}v\in\operatorname{Im}(\Omega^n\wedge\overline{\Omega}^n)$, which is impossible, because $\Omega^n\wedge\overline{\Omega}^n$ is non-degenerate. Then $T^{0,1}_\Omega(M)\oplus\overline{T^{0,1}_\Omega(M)}=TM\otimes_{\mathbb{R}}\mathbb{C}$, defining an almost complex structure I_Ω. Its integrabiliuty immediately follows from Proposition 1. ∎

Period map for holomorphically symplectic manifolds

DEFINITION: Let (M, I, Ω) be a holomorphically symplectic manifold, and CSymp the space of all C-symplectic forms. The quotient CTeich $:= \frac{\text{CSymp}}{\text{Diff}_0}$ is called **the holomorphically symplectic Teichmüller space**, and the map CTeich $\longrightarrow H^2(M, \mathbb{C})$ taking (M, I, Ω) to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$ **the holomorphically symplectic period map**.

We want to prove that **the period map is locally an embedding.** This is immediately implied by the following version of Moser's lemma.

THEOREM: Let (M, I_t, Ω_t), $t \in [0, 1]$ be a family of C-symplectic forms on a compact manifold. Assume that the cohomology class $[\Omega_t] \in H^2(M, \mathbb{C})$ is constant, and $H^{0,1}(M, I_t) = 0$, where $H^{0,1}(M, I_t) = H^1(M, \mathcal{O}_{(M,I_t)})$ is cohomology of the sheaf of holomorphic functions. Then **there exists a smooth family of diffeomorphisms $V_t \in \text{Diff}_0(M)$, such that $V_t^* \Omega_0 = \Omega_t$.**

Proof: Later in this course.

Local Torelli theorem

REMARK: In real dimension 4, C-symplectic form **is a pair ω_1, ω_2 of symplectic forms which satisfy $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$.**

THEOREM: Let (M, I, Ω) be a complex holomorphically symplectic surface with $H^{0,1}(M) = 0$, that is, a K3 surface. Consider the period map $\operatorname{Per}: \operatorname{CTeich} \longrightarrow H^2(M, \mathbb{C})$ taking (M, I, Ω) to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$. **Then** Per **is a local diffeomorpism** of CTeich to the **period space** $Q := \{v \in H^2(M, \mathbb{C}) \quad | \quad \int_M v \wedge v = 0, \int_M v \wedge \overline{v} > 0\}$.

Proof: Later in this course.

A caution: CTeich is smooth, but non-Hausdorff. The non-Hausdorff points are well understood and correspond to the partition of the "positive cone" $\{v \in H^{1,1}_I(M, \mathbb{R}) \quad | \quad \int_M v \wedge v > 0\}$ onto "Kähler chambers" (to be explained later).