K3 surfaces

lecture 6: C-symplectic structures

Misha Verbitsky

IMPA, sala 236

October 24, 2022, 15:30

Complex manifolds (reminder)

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

REMARK: The commutator defines a $C^\infty M$-linear map $N := \Lambda^2(T^{1,0}) \longrightarrow T^{0,1}M$, called **the Nijenhuis tensor** of I. **One can represent N as a section of $\Lambda^{2,0}(M) \otimes T^{0,1}M$.**

Exercise: Prove that $\mathbb{C}P^n$ **is a complex manifold,** in the sense of the above definition.

Holomorphic symplectic form (reminder)

DEFINITION: Holomorphic symplectic form on an almost complex manifold (M, I) is a non-degenerate differential 2-form $\omega \in \Lambda^2(M, \mathbb{C})$ satisfying $d\omega = 0$ and $\Omega(Ix, y) = \sqrt{-1}\,\Omega(x, y)$.

REMARK: This is the same as to say that $\Omega(X, \cdot) = 0$ **for all** $X \in T^{1,0}(M)$**, and** Ω **is non-degenerate on** $T^{0,1}(M)$**.**

DEFINITION: Let Ω be a differential form on M. The **kernel**, or **the null-space** $\ker(\Omega) \subset TM$ of Ω is the space of all vector fields $X \in TM$ such that the contraction $\Omega \lrcorner X$ vanishes.

Holomorphic symplectic form and integrability (reminder)

Theorem 1: Let Ω be a p-form on a smooth manifold, and $B = \ker\Omega$. **Then $[B,B]\subset B$,** that is, **the distribution B is integrable.**

Proof. Step 1: Cartan's formula implies that $\operatorname{Lie}_X(\Omega) = d(i_X(\Omega)) + i_x(d\Omega) = 0$, **hence $\operatorname{Lie}_X(\Omega) = 0$ for any $X\in B$.**

Step 2: Let $X, X_1 \in B$, and $X_2, ..., X_p$ any vector fields. Cartan formula gives

$$\operatorname{Lie}_X(\Omega)(X_1, ..., X_p) = \operatorname{Lie}_X(\Omega(X_1, ..., X_p)) - \sum_{i=1}^{p} \Omega(X_1, ..., [X, X_i], ...X_p).$$

All terms of this sum, except $\Omega([X, X_1], X_2, ..., X_p)$, vanish, because $X_1 \in B$ and $\operatorname{Lie}_X(\Omega) = 0$. **This gives $\Omega([X, X_1], X_2, ..., X_p) = 0$ for all $X_2, ..., X_p$.** Therefore, $[X, X_1] \in B$. ■

COROLLARY: Let (M, I) be an almost complex manifold admitting a holomorphic symplectic form. **Then I is integrable.**

Proof: Indeed, $\ker\Omega = T^{0,1}(M)$. ■

C-symplectic structures

DEFINITION: Let M be a smooth $4n$-dimensional manifold. A complex-valued form Ω on M is called **almost C-symplectic** if $\Omega^{n+1}=0$ and $\Omega^n\wedge\overline{\Omega}^n$ is a non-degenerate volume form and **C-symplectic** if it is also closed.

THEOREM: Let $\Omega\in\Lambda^2(M,\mathbb{C})$ be an almost C-symplectic form, and $T^{0,1}_\Omega(M)$ be equal to $\ker\Omega$, where

$$\ker\Omega:=\{v\in TM\otimes\mathbb{C}\ \ |\ \ \Omega\lrcorner v=0\}.$$

Then $T^{0,1}_\Omega(M)\oplus\overline{T^{0,1}_\Omega(M)}=TM\otimes_{\mathbb{R}}\mathbb{C}$, hence **the sub-bundle $T^{0,1}_\Omega(M)$ defines an almost complex structure I_Ω on M.** If, in addition, **Ω is closed, I_Ω is integrable, and Ω is holomorphically symplectic on (M,I_Ω).**

Proof: Rank of Ω is $2n$ because $\Omega^{n+1}=0$ and $\operatorname{Re}\Omega$ is non-degenerate. Then $\ker\Omega\oplus\overline{\ker\Omega}=T_{\mathbb{C}}M$. The relation $[T^{0,1}_\Omega(M),T^{0,1}_\Omega(M)]\subset T^{0,1}_\Omega(M)$ follows from Theorem 1. ■

Holomorphically symplectic Teichmüller space

DEFINITION: Let CSymp be the space of all C-symplectic forms on a manifold M, equipped with the C^∞-topology, and Diff_0 the connected component of the group of diffeomorphisms. The **holomorphically symplectic Teichmüller space** CTeich is the quotient $\frac{\mathrm{CSymp}}{\mathrm{Diff}_0}$.

REMARK: Recall that **the mapping class group** of a manifold M is the group $\Gamma := \frac{\mathrm{Diff}}{\mathrm{Diff}_0}$ of connected components of $\mathrm{Diff}(M)$.

REMARK: The quotient CTeich/Γ is identified with the set of all holomorphically symplectic structures on M up to isomorphism.

Period map for holomorphically symplectic manifolds (reminder)

DEFINITION: Let (M, I, Ω) be a holomorphically symplectic manifold, and CSymp the space of all C-symplectic forms. The quotient $\operatorname{CTeich} := \frac{\operatorname{CSymp}}{\operatorname{Diff}_0}$ is called **the holomorphically symplectic Teichmüller space**, and the map $\operatorname{CTeich} \longrightarrow H^2(M, \mathbb{C})$ taking (M, I, Ω) to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$ **the holomorphically symplectic period map**.

THEOREM: (Local Torelli theorem, due to Bogomolov)
Let (M, I, Ω) be a complex holomorphically symplectic surface with $H^{0,1}(M) = 0$, that is, a K3 surface. Consider the period map $\operatorname{Per} : \operatorname{CTeich} \longrightarrow H^2(M, \mathbb{C})$ taking (M, I, Ω) to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$. **Then Per is a local diffeomorpism** of CTeich to the **period space** $Q := \{v \in H^2(M, \mathbb{C}) \ \ | \ \ \int_M v \wedge v = 0, \int_M v \wedge \overline{v} > 0\}$.

Proof: Later in this course.

A caution: CTeich is smooth, but non-Hausdorff. The non-Hausdorff points are well understood and correspond to the partition of the "positive cone" $\{v \in H^{1,1}_I(M, \mathbb{R}) \ \ | \ \ \int_M v \wedge v > 0\}$ onto "Kähler chambers" (to be explained later).

C-symplectic structures on surfaces

CLAIM: In real dimension 4, a C-symplectic structure **is determined by a pair $\omega_1 = \operatorname{Re}\Omega, \omega_2 = \operatorname{Im}\Omega$ of symplectic forms which satisfy $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$.**

Proof: Let Ω be a C-symplectic form, $\omega_1 = \operatorname{Re}\Omega$ and $\omega_2 = \operatorname{Im}\Omega$. Then $\Omega \wedge \Omega = \omega_1^2 + \omega_2^2 + 2\sqrt{-1}\,\omega_1 \wedge \omega_2 = 0$, hence $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$. The form $\Omega \wedge \overline{\Omega} = \omega_1^2 + \omega_2^2$ is non-degenerate, hence $\omega_1^2 = \omega_2^2$ is non-degenerate.

Conversely, if $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$, we have $\Omega \wedge \Omega = 0$, and $\Omega \wedge \overline{\Omega} = \omega_1^2 + \omega_2^2$ is non-degenerate if ω_i is non-degenerate. ■

REMARK: For K3 surface, the local Torelli theorem would imply that the period map $\operatorname{Per}: \operatorname{CTeich} \longrightarrow H^2(M, \mathbb{C})$ which takes ω_1, ω_2 to their cohomology classes which satisfy $[\omega_1]^2 = [\omega_2]^2$ and $[\omega_1] \wedge [\omega_2] = 0$ **is locally a diffeomorphism.**

Compare this with Moser theorem (Lecture 4)