K3 surfaces

lecture 8: dd^2-lemma.

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dd^c-lemma

THEOREM: Let η be a form on a compact Kähler manifold, satisfying one of the following conditions.
(1). η is an exact (p,q)-form. (2). η is d-exact, d^c-closed.
(3). η is ∂-exact, $\overline{\partial}$-closed.
Then $\eta \in \operatorname{im} dd^c = \operatorname{im} \partial\overline{\partial}$**.**

Proof: Notice immediately that in all three cases η **is closed and orthogonal to the kernel of** Δ, hence its cohomology class vanishes. Indeed, $\ker \Delta$ is orthogonal to the image of $\partial, \overline{\partial}$ and d. Since η is exact, it lies in the image of Δ. Operator $G_\Delta := \Delta^{-1}$ is defined on $\operatorname{im}\Delta = \ker \Delta^\perp$ and commutes with d, d^c.

In case (1), η is d-exact, and $I(\eta) = (\sqrt{-1}\,)^{p-q}\eta$ is d-closed, hence η is d-exact, d^c-closed like in (2). Then $\eta = d\alpha$, where $\alpha := G_\Delta d^*\eta$. Since G_Δ and d^* commute with d^c, the form α is d^c-closed; since it belongs to $\operatorname{im}\Delta = \operatorname{im} G_\Delta$, it is d^c-exact, $\alpha = d^c\beta$ which gives $\eta = dd^c\beta$.

In case (3), we have $\eta = \partial\alpha$, where $\alpha := G_\Delta \partial^*\eta$. Since G_Δ and ∂^* commute with $\overline{\partial}$, the form α is $\overline{\partial}$-closed; since it belongs to $\operatorname{im}\Delta$, it is $\overline{\partial}$-exact, $\alpha = \overline{\partial}\beta$ which gives $\eta = \partial\overline{\partial}\beta$. ■

Massey products

As an application of dd^c-lemma, I would prove a theorem about topology of compact Kähler manifolds.

Let $a, b, c \in \Lambda^*(M)$ be closed forms on a manifold M with cohomology classes $[a], [b], [c]$ satisfying $[a][b] = [b][c] = 0$, and $\alpha, \gamma \in \Lambda^*(M)$ forms which satisfy $d(\alpha) = a \wedge b$, $d(\gamma) = b \wedge c$. Denote by $L_{[a]}, L_{[c]} : H^*(M) \longrightarrow H^*(M)$ the operation of multiplication by the cohomology classes $[a], [c]$.

Then $\alpha \wedge c - a \wedge \gamma$ is a closed form, and its cohomology class is well-defined modulo $\operatorname{im} L_{[a]} + \operatorname{im} L_{[c]}$.

DEFINITION: Cohomology class $\alpha \wedge c - a \wedge \gamma$ is called **Massey product of a, b, c.**

PROPOSITION: On a Kähler manifold, Massey products vanish.

Proof: Let a, b, c be harmonic forms of pure Hodge type, that is, of type (p, q) for some p, q. Then ab and bc are exact pure forms, hence $ab, bc \in \operatorname{im} dd^c$ by dd^c-lemma. This implies that $\alpha := d^* G_\Delta(ab)$ and $\gamma := d^* G_\Delta(bc)$ are d^c-exact. Therefore $\mu := \alpha \wedge c - a \wedge \gamma$ is a d^c-exact, d-closed form. **Applying dd^c-lemma again, we obtain that μ is dd^c-exact, hence its cohomology class vanish.**

■

Heisenberg group

REMARK: *In the class, we constructed this space explicitly as a cell complex, without using the Lie algebra, and computed the Massey product in its cohomology. Here are the notes taken from a lecture given elsewhere, which explain the same construction in a different, more algebraic way.*

DEFINITION: The **Heisenberg group** G group of strictly upper triangular matrices (3x3),

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

The integer Heisenberg group $G_{\mathbb Z}$ is the same group with integer entries. The **Heisenberg nilmanifold** is $G/G_{\mathbb Z}$. The Heisenberg nilmanifold **is fibered over the torus T^2 with the fiber S^1** (it is a non-trivial principal S^1-bundle). This fibration corresponds to the exact sequence

$$\{e\} \longrightarrow \mathbb Z \longrightarrow G_{\mathbb Z} \longrightarrow \mathbb Z^2 \longrightarrow \{e\}$$

where $\mathbb Z$ is the center.

Massey products in Heisenberg nilmanifold

CLAIM: Masey products on $G/G_{\mathbb Z}$ are non-zero.

Proof. Step 1: G acts on $\Lambda^*(G)$ from the right. It is not hard to see that the all cohomology classes on $G/G_{\mathbb Z}$ can be represented by right G-invariant forms, and, moreover, **the cohomology of $G/G_{\mathbb Z}$ is equal to the cohomology of the complex of right-G-invariant forms on G.**

Step 2: This is the same complex as **the Chevalley-Eilenberg complex** for the Lie algebra $\mathfrak g$ of G: $0 \longrightarrow \Lambda^1(\mathfrak g^*) \xrightarrow{d} \Lambda^2(\mathfrak g^*) \xrightarrow{d}$ ... with the differential in the first term $d: \mathfrak g^* \longrightarrow \Lambda^2(\mathfrak g^*)$ dual to the commutator. We extend this differential to $\Lambda^*(\mathfrak g^*)$ by the Leibniz rule. The corresponding cohomology is called **the Lie algebra cohomology** and denoted by $H^*(\mathfrak g)$.

Step 3: Let a, b, t be the basis in $\mathfrak g$, with the only non-trivial commutator $[a, b] = t$, and α, β, τ the dual basis in $\mathfrak g^*$, with the only non-trivial differential $d\tau = \alpha\wedge\beta$. This gives a basis $\alpha\wedge\beta$, $\alpha\wedge\tau$, $\beta\wedge\tau$ in $\Lambda^2(\mathfrak g^*)$, with $d\big|_{\Lambda^2\mathfrak g^*} = 0$, giving $\operatorname{rk} H^1(G/G_{\mathbb Z}) = 2$ and $\operatorname{rk} H^2(G/G_{\mathbb Z}) = 2$.

Step 4: Let $M(\alpha, \beta, \alpha)$ denote the Massey product of α, β, α. Since $\alpha\wedge\beta = d\tau$, $M(\alpha, \beta, \alpha) = \tau\wedge\alpha - \alpha\wedge\tau = 2\tau\wedge\alpha$. The image of $L_\alpha: H^1(\mathfrak g) \longrightarrow H^2(\mathfrak g)$ is generated by $\alpha\wedge\beta$, **hence $M(\alpha, \beta, \alpha)$ is non-zero modulo $\operatorname{im} L_\alpha$.** ■