K3 surfaces

lecture 9: local Torelli theorem. Surjectivity of the period map.

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C-symplectic structures (reminder)

DEFINITION: Let M be a smooth $4n$-dimensional manifold. A closed complex-valued form Ω on M is called **C-symplectic** if $\Omega^{n+1}=0$ and $\Omega^n\wedge\overline{\Omega}^n$ is a non-degenerate volume form.

THEOREM: Let $\Omega\in\Lambda^2(M,\mathbb{C})$ be a C-symplectic form, and $T^{0,1}_\Omega(M)$ be equal to $\ker\Omega$, where

$$\ker\Omega := \{v\in TM\otimes\mathbb{C}\ \ |\ \ \Omega\lrcorner v=0\}.$$

Then $T^{0,1}_\Omega(M)\oplus\overline{T^{0,1}_\Omega(M)}=TM\otimes_{\mathbb{R}}\mathbb{C}$, hence **the sub-bundle $T^{0,1}_\Omega(M)$ defines an almost complex structure I_Ω on M.** If, in addition, **Ω is closed, I_Ω is integrable, and Ω is holomorphically symplectic on (M,I_Ω).**

Proof: Rank of Ω is $2n$ because $\Omega^{n+1}=0$ and $\operatorname{Re}\Omega$ is non-degenerate. Then $\ker\Omega\oplus\overline{\ker\Omega}=T_{\mathbb{C}}M$. The relation $[T^{0,1}_\Omega(M),T^{0,1}_\Omega(M)]\subset T^{0,1}_\Omega(M)$ follows from Theorem 1, Lecture 6. ■

Holomorphically symplectic Teichmüller space (reminder)

DEFINITION: Let CSymp be the space of all C-symplectic forms on a manifold M, equipped with the C^∞-topology, and Diff_0 the connected component of the group of diffeomorphisms. The **holomorphically symplectic Teichmüller space** CTeich is the quotient $\frac{\mathsf{CSymp}}{\mathsf{Diff}_0}$.

REMARK: Recall that **the mapping class group** of a manifold M is the group $\Gamma := \frac{\mathsf{Diff}}{\mathsf{Diff}_0}$ of connected components of $\mathsf{Diff}(M)$.

REMARK: The quotient CTeich/Γ **is identified with the set of all holomorphically symplectic structures on M up to isomorphism.**

Period map for holomorphically symplectic manifolds (reminder)

DEFINITION: Let (M, I, Ω) be a holomorphically symplectic manifold, and CSymp the space of all C-symplectic forms. The quotient $\text{CTeich} := \frac{\text{CSymp}}{\text{Diff}_0}$ is called **the holomorphically symplectic Teichmüller space**, and the map $\text{CTeich} \longrightarrow H^2(M, \mathbb{C})$ taking (M, I, Ω) to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$ **the holomorphically symplectic period map**.

THEOREM: (Local Torelli theorem, due to Bogomolov)
Let (M, I, Ω) be a complex, Kähler, holomorphically symplectic surface with $H^{0,1}(M) = 0$, that is, a K3 surface. Consider the period map

$$\text{Per}: \ \text{CTeich} \longrightarrow H^2(M, \mathbb{C})$$

taking (M, I, Ω) to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$. **Then Per is a local diffeomorpism** of CTeich to the **period space**

$$Q := \left\{ v \in H^2(M, \mathbb{C}) \quad | \quad \int_M v \wedge v = 0, \int_M v \wedge \overline{v} > 0 \right\}.$$

Proof: Surjectivity: later today. Injectivity: lecture 7. ■

dd^c-lemma (reminder)

THEOREM: Let η be a form on a compact Kähler manifold, satisfying one of the following conditions.
(1). η is an exact (p,q)-form. (2). η is d-exact, d^c-closed.
(3). η is ∂-exact, $\overline{\partial}$-closed.
Then $\eta \in \operatorname{im} dd^c = \operatorname{im} \partial\overline{\partial}$.

Proof: Notice immediately that in all three cases η **is closed and orthogonal to the kernel of Δ**, hence its cohomology class vanishes. Indeed, $\ker\Delta$ is orthogonal to the image of $\partial, \overline{\partial}$ and d. Since η is exact, it lies in the image of Δ. Operator $G_\Delta := \Delta^{-1}$ is defined on $\operatorname{im}\Delta = \ker\Delta^\perp$ and commutes with d, d^c.

In case (1), η is d-exact, and $I(\eta) = (\sqrt{-1}\,)^{p-q}\eta$ is d-closed, hence η is d-exact, d^c-closed like in (2). Then $\eta = d\alpha$, where $\alpha := G_\Delta d^*\eta$. Since G_Δ and d^* commute with d^c, the form α is d^c-closed; since it belongs to $\operatorname{im}\Delta = \operatorname{im} G_\Delta$, it is d^c-exact, $\alpha = d^c\beta$ which gives $\eta = dd^c\beta$.

In case (3), we have $\eta = \partial\alpha$, where $\alpha := G_\Delta\partial^*\eta$. Since G_Δ and ∂^* commute with $\overline{\partial}$, the form α is $\overline{\partial}$-closed; since it belongs to $\operatorname{im}\Delta$, it is $\overline{\partial}$-exact, $\alpha = \overline{\partial}\beta$ which gives $\eta = \partial\overline{\partial}\beta$. ■

Deformation of C-symplectic structures

Let Ω be a holomorphically symplectic form on a complex surface, and $u \in \Lambda^{1,1} + \Lambda^{0,2}$. Then $\Omega + u$ is C-symplectic if and only if $du = 0$ and

$$(\Omega + \eta)^2 = \eta \wedge \eta - u^{0,2} \wedge \Omega = 0 \quad (*)$$

where $\eta = u^{1,1}$. Denote by $\Lambda_\Omega : \Lambda^{p,q}(M) \longrightarrow \Lambda^{p,q-2}(M)$ the operator of convolution with the $(2,0)$-bivector dual to Ω; it is inverse to L_Ω, where $L_\Omega(x) = x \wedge \Omega$. **The equation (*) is equivalent to $\Lambda_\Omega(\eta \wedge \eta) = -u^{0,2}$.**

Then $du = 0$ is equivalent to

$$\partial\eta = 0, \quad \overline{\partial}(\Lambda_\Omega(\eta \wedge \eta)) = 0, \quad \overline{\partial}\eta = \partial(\Lambda_\Omega(\eta \wedge \eta)).$$

The second equation is automatic, because Λ_Ω commutes with $\overline{\partial}$, since the bivector dual to Ω is holomorphic, and $\overline{\partial}\eta \wedge \eta = 0$ because it is a 5-form. We proved the following theorem.

THEOREM: Let Ω be a holomorphically symplectic form on a complex surface, and $u \in \Lambda^{1,1} + \Lambda^{0,2}$. Let $\eta := u^{1,1}$. **Then $\Omega + u$ is C-symplectic if and only if $\partial\eta = 0$, $u^{2,0} = \Lambda_\Omega(\eta \wedge \eta)$, and $\overline{\partial}\eta = \partial u^{2,0}$.** ■

Recursive solutions of Maurer-Cartan equation

REMARK: The equation $\overline{\partial}\eta = \partial(\Lambda_\Omega(\eta \wedge \eta))$ ("symplectic Maurer-Cartan equation") is a form of Maurer-Cartan equation known in deformation theory. **We will solve it in the same way as the usual Maurer-Cartan: recursively.**

DEFINITION: Let $\eta_i \in \Lambda^{1,1}(M)$, $i = 0, 1, ...$ be (1,1)-forms on a holomorphic symplectic surface. We say that $\{\eta_i\}$ is **a recursive solution of the symplectic Maurer-Cartan equation**, if

(i) the series $\sum_i \eta_i t^i$ **converges absolutely for** $|t| < \varepsilon$

(ii) η_i **are** ∂**-closed and** ∂**-exact for** $i > 0$**.**

(iii) $\overline{\partial}\eta_n = \sum_{i+j=n-1} \partial(\Lambda_\Omega(\eta_i \wedge \eta_j))$ **for all** n**.**

From recursive solutions to solutions

PROPOSITION: Let $\{\eta_i\}$ be a recursive solution of the symplectic Maurer-Cartan equation, and $|t| < \varepsilon$. Consider the cohomology class $[\eta_0] \in H^{1,1}_\partial(M) = H^{1,1}(M)$ (here **we identify the Dolbeault cohomology and the Hodge cohomology of M).** Let $\eta := \sum_i \eta_i t^i$. **Then $\Omega_\eta := \Omega + \eta - \Lambda_\Omega(\eta \wedge \eta)$ is $\mathbb{C}$-symplectic, for t sufficiently small, and cohomologous to $[\Omega + \eta_0 - L_\Omega^{-1}(\eta_0 \wedge \eta_0)]$,** where L_Ω is the multiplication by Ω acting on cohomology.

Proof. Step 1: Since Λ_Ω commutes with $\bar\partial$, we have

$$\bar\partial \Lambda_\Omega(\eta \wedge \eta) = 2\Lambda_\Omega(\bar\partial\eta \wedge \eta) = 0,$$

because $\bar\partial\eta \wedge \eta$ is a (2,3)-form. Also, $\partial\eta = 0$, because all η_i are ∂-closed. Then

$$d(\Omega_\eta) = \bar\partial\eta - \partial\Lambda_\Omega(\eta \wedge \eta) = \sum_n t^n \left(\bar\partial\eta_n - \sum_{i+j=n-1} \partial(\Lambda_\Omega(\eta_i \wedge \eta_j)\right) = 0.$$

Step 2: Since $\Omega_\eta^2 = -\Omega \wedge \Lambda_\Omega(\eta \wedge \eta) + \eta \wedge \eta$ and $\Omega \wedge (\Lambda_\Omega(\eta \wedge \eta)) = \eta \wedge \eta$, we have $\Omega_\eta^2 = 0$. For η sufficiently small, $\Omega_\eta \wedge \overline{\Omega}_\eta$ is non-degenerate, because $\Omega \wedge \overline{\Omega}$ is non-degenerate. **Therefore, Ω_η is C-symplectic.** It remains only **to express the cohomology class of Ω_η through $[\eta_0] \in H^{1,1}_\partial(M)$.**

From recursive solutions to solutions (2)

PROPOSITION: Let $\{\eta_i\}$ be a recursive solution of the symplectic Maurer-Cartan equation as above, and $\eta := \sum_i \eta_i t^i$. **Then $\Omega_\eta := \Omega + \eta - \Lambda_\Omega(\eta\wedge\eta)$ is $\mathbb{C}$-symplectic, for t sufficiently small, and cohomologous to $[\Omega + \eta_0 - L_\Omega^{-1}(\eta_0\wedge\eta_0)]$,** where L_Ω is the multiplication by Ω acting on cohomology.

Step 1-2: Ω_η is C-symplectic.

Step 3: The (1,2)-form $d(\Lambda_\Omega(\eta\wedge\eta)) = \partial(\Lambda_\Omega(\eta\wedge\eta) = \overline{\partial}\eta$ is ∂-exact and $\overline{\partial}$-exact, hence it is $\partial\overline{\partial}$-exact. Therefore, **there exists a (0,1)-form α such that $\partial(\Lambda_\Omega(\eta\wedge\eta)) = \partial\overline{\partial}\alpha$.** Then, the (0,2)-form $\beta := \Lambda_\Omega(\eta\wedge\eta) - \overline{\partial}\alpha$ is closed. We obtain that all Hodge components of $\Omega_\eta - d\alpha$ are closed: indeed, $d(\Omega_\eta - d\alpha) = 0$ and the form $(\Omega_\eta - d\alpha)^{0,2} = \beta$ is $\overline{\partial}$ and ∂-closed. Therefore, the cohomology class $[\Omega_\eta]$ is equal to the sum of the Dolbeault classes of Ω, $\eta^{1,1}$ and β (this form is antiholomorphic). The de Rham cohomology class of $\eta - \partial\alpha$ is equal to its Dolbeault class $[\eta_0]$, and the de Rham cohomology class $[\beta] = const\cdot[\overline{\Omega}]$ of β is uniquely determined from

$$0 = [\Omega_\eta]^2 = ([\Omega] + [\eta_0] + [\beta])^2 = [\eta_0]^2 + 2[\Omega]\wedge[\beta].$$

■

From recursive solutions to deformations

PROPOSITION: Let $\{\eta_i\}$ be a recursive solution of the symplectic Maurer-Cartan equation, and $|t| < \varepsilon$. Consider the cohomology class $[\eta_0] \in H^{1,1}_{\partial}(M) = H^{1,1}(M)$ (here **we identify the Dolbeault cohomology and the Hodge cohomology of M).** Let $\eta := \sum_i \eta_i t^i$. **Then $\Omega_\eta := \Omega + \eta - \Lambda_\Omega(\eta \wedge \eta)$ is $\mathbb{C}$-symplectic, for t sufficiently small, and cohomologous to $[\Omega + \eta_0 - L_\Omega^{-1}(\eta_0 \wedge \eta_0)]$,** where L_Ω is the multiplication by Ω acting on cohomology.

Therefore, the local surjectivity of the period map is implied by the following

THEOREM: Let (M, Ω) be a compact holomorphic symplectic surface, and $\eta_0 \in \Lambda^{1,1}(M)$ a closed (1,1)-form. **Then there exists a recursive solution $\{\eta_i\} \subset \Lambda^{1,1}(M)$ of the symplectic Maurer-Cartan equation.**

Proof: Later today.

Holomorphically symplectic Schouten bracket

DEFINITION: Let $a, b \in \Lambda^{*,1}$ be $(*,1)$-forms on a holomorphically symplectic manifold (M,Ω), and

$$\{a,b\} := \delta(a \wedge b) - \delta(a) \wedge b - (-1)^{\tilde{a}-1} a \wedge \delta(b),$$

where $\delta = [\partial, \Lambda_\Omega]$. We call $\{\cdot,\cdot\}$ **the holomorphically symplectic Schouten bracket**.

REMARK: For (1,1)-forms, $\{a,b\} := \partial(\Lambda_\Omega(a \wedge b) - \Lambda_\Omega(\partial(a \wedge b))$.

REMARK: Clearly, **the Schouten bracket is graded commutative:**

$$\{a,b\} - (-1)^{\tilde{a}\tilde{b}}\{b,a\} = 0.$$

THEOREM 1: The holomorphically symplectic Schouten bracket **satisfies the odd graded Jacobi identity**

$$\{a,\{b,c\}\} = \{\{a,b\},c\} + (-1)^{(\tilde{a}-1)(\tilde{b}-1)}\{b,\{a,c\}\}.$$

Proof: Lecture 9 (next Monday). ■

The third commutator

Whenever a bracket $\{\cdot,\cdot\}$ is supercommutative and satisfies the graded Jacobi identity, **the third commutator of a vector with itself vanishes.**

COROLLARY: For any $v \in \Lambda^{*,1}$, **one has** $\{v,\{v,v\}\} = 0$.

Proof: If v is odd, $\{v,v\} = 0$ because the Schouten bracket is graded commutative. If v is even, the odd graded Jacobi identity gives

$$\{v,\{v,v\}\} = \{\{v,v\},v\} - \{v,\{v,v\}\},$$

hence $2\{v,\{v,v\}\} = \{\{v,v\},v\} = -\{v,\{v,v\}\}$, implying $3\{v,\{v,v\}\} = 0$. ■

Solving the symplectic Maurer-Cartan equation

THEOREM: Let (M,Ω) be a compact holomorphic symplectic surface, and $\eta_0 \in \Lambda^{1,1}(M)$ a closed (1,1)-form. **Then there exists a recursive solution $\{\eta_i\} \subset \Lambda^{1,1}(M)$ of the symplectic Maurer-Cartan equation**

$$\overline{\partial}\eta_n = \sum_{i+j=n-1} \{\eta_i, \eta_j\}.$$

Proof. Step 1: Suppose that $\eta_0, ..., \eta_{n-1} \in \Lambda^{1,1}(M)$ are already found, and satisfy $\overline{\eta}_k = \sum_{i+j=k-1}\{\eta_i, \eta_j\}$, $\partial\eta_i = 0$ for all $k = 0, ..., n-1$. For any ∂-closed $a, b \in \Lambda^{*,1}(M)$, we have

$$\overline{\partial}\{a,b\} = \overline{\partial}\{a,b\} = \overline{\partial}\partial(\Lambda_\Omega(a \wedge b) = -\partial(\Lambda_\Omega(\overline{\partial}a \wedge b) + (-1)^{\tilde{a}}\partial(\Lambda_\Omega(a \wedge \overline{\partial}b),$$

because $\overline{\partial}$ commutes with Λ_Ω and anticommutes with ∂. Taking $a = \eta_k, b = \eta_l$, $k, l < n$ we obtain

$$\overline{\partial}\{\eta_k, \eta_l\} = \left\{\sum_{i+j=k-1} \{\eta_i, \eta_j\}, \eta_l\right\} + \left\{\eta_k, \sum_{i+j=l-1} \{\eta_i, \eta_j\}\right\}.$$

Solving the symplectic Maurer-Cartan equation (2)

THEOREM: Let (M,Ω) be a compact holomorphic symplectic surface, and $\eta_0 \in \Lambda^{1,1}(M)$ a closed (1,1)-form. **Then there exists a recursive solution $\{\eta_i \in \Lambda^{1,1}(M)\}$ of the symplectic Maurer-Cartan equation $\overline{\partial}\eta_n = \sum_{i+j=n-1}\{\eta_i,\eta_j\}$.**

Proof. Step 1: $\overline{\partial}\{\eta_k,\eta_l\} = \left\{\sum_{i+j=k-1}\{\eta_i,\eta_j\},\eta_l\right\} + \left\{\eta_k,\sum_{i+j=l-1}\{\eta_i,\eta_j\}\right\}.$

Step 2: Let $\theta_n := \sum_{i=0}^{n-1} t^i\eta_i$. Then Theorem 1 implies that

$$\overline{\partial}\{\theta_n,\theta_n\} = \sum_{m=0}^{n-1}\sum_{k+l=m} t^{k+l}\overline{\partial}\{\eta_k,\eta_l\} = \sum_{m=0}^{n-1}\sum_{p+q+r=m-1} t^{p+q+r}\Big(\{\{\eta_p,\eta_q\},\eta_r\} + \{\eta_p,\{\eta_q,\eta_r\}\}\Big) = \{\theta_n,\{\theta_n,\theta_n\}\} + \{\{\theta_n,\theta_n\},\theta_n\} = 0.$$

This gives $\overline{\partial}\left(\sum_{i+j=n-1}\{\eta_i,\eta_j\}\right) = 0$.

Step 3: Since $\sum_{i+j=n-1}\{\eta_i,\eta_j\} = \sum_{i+j=n-1}\partial\Lambda_\Omega(\eta_i\wedge\eta_j)$ is ∂-closed and ∂-exact, this form is $\overline{\partial}\partial$-exact: $\overline{\partial}\partial\beta = \sum_{i+j=n-1}\{\eta_i,\eta_j\}$. Take $\eta_n := \partial\beta$; this form is ∂-exact **and satisfies the equation $\overline{\partial}\eta_n = \sum_{i+j=n-1}\{\eta_i,\eta_j\}$. It remains to make sure that the power series $\sum_{i=0}^{\infty} t^i\eta_i$ converges for t sufficiently small.**

Solving the symplectic Maurer-Cartan equation (3)

Step 4: We will use the operator $G_\Delta \overline{\partial}^*$ to invert $\overline{\partial}$; this was justified earlier using the Hodge theory. This allows us to take $\eta_n = G_\Delta \overline{\partial}^* (\sum_{i+j=n-1} \{\eta_i, \eta_j\})$. **It remains to check the convergence of the power series $\sum_{i=0}^\infty t^i \eta_i$,** for $|\eta_0|$ sufficiently small. We supply the space $\Lambda^*(M)$ with a Hilbert norm L^2_r associated with the integral of sum of first r derivatives. Since Δ is elliptic, its inverse G_Δ is diagonal with bounded eigenvalues. Using the same arguments, it is possible to show that the operator $G_\Delta \overline{\partial}^* \partial$ has bounded operator norm.

Step 5: Applying the Cauchy-Schwarz inequality to deduce $|a \wedge b| < A|a||b|$, we obtain

$$|\eta_n| = \left| G_\Delta \overline{\partial}^* \left(\sum_{i+j=n-1} \{\eta_i, \eta_j\} \right) \right| \leqslant AC \sum_{i+j=n-1} |\eta_i||\eta_j|$$

where C is the operator norm of $G_\Delta \overline{\partial}^* \partial$. Iterating this estimate, we obtain $|\eta_n| \leqslant C^n S_n |\eta_0|^n$, where S_n is the n-th Catalan number (the number of ways of putting n brackets into a sum of $n+1$ terms). Catalan numbers can be expressed as $S_n = \frac{2n!}{n!(n+1)!} \leqslant 2^n$, which gives $|\eta_n| \leqslant C^n 2^n |\eta_0|^n$. **Then the series $\sum t^i |\eta_i|$ converges absolutely for some $t > 0$.** ■