K3 surfaces

lecture 10: Schouten brackets and holomorphic symplectic form.

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Derivations on $\Lambda^{0,*}(M)$

DEFINITION: Let $a \in \Lambda^{0,r} \otimes T^{1,0}(M)$, where M is a complex manifold. Using local coordinates, we can write a locally as a sum of coordinate monomials $\sum_J d\bar{z}_J \otimes X_J$, where $X_J \in T^{1,0}(M)$ is a vector field, and $d\bar{z}_J = d\bar{z}_{j_1} \wedge ... \wedge d\bar{z}_{j_r}$ a coordinate monomial. **Denote by $\operatorname{Lie}_a(u) : \Lambda^{p,q} \longrightarrow \Lambda^{p,q+r}$ a derivation which takes a form $f_{KL} dz_K \wedge d\bar{z}_L$ to $\sum_J \operatorname{Lie}_{X_J}(f_{KL}) \wedge d\bar{z}_J \wedge dz_K \wedge d\bar{z}_L$.**

REMARK: Clearly, Lie_a **is a superderivation which vanishes on antiholomorphic forms.**

CLAIM: Any derivation $\delta : \Lambda^{0,*} \longrightarrow \Lambda^{0,*+p}$ vanishing on antiholomorphic forms **is equal to Lie_a for an appropriate $a \in \Lambda^{0,p} \otimes T^{1,0}(M)$.**

Proof. Step 1: A superderivation **is determined by its restriction to any set of multiplicative generators.**

Step 2: Since δ is a derivation, its restriction to $C^\infty M$ defines an $\Lambda^{0,p}(M)$-valued vector field $a \in \Lambda^{0,p} \otimes TM$. Since a vanishes on antiholomorphic functions, it belogs to $\Lambda^{0,p} \otimes T^{1,0}(M)$. Now, **$\delta$ and Lie_a are derivations which agree on antiholomorphic forms and on functions,** and antiholomorphic forms and functions generate $\Lambda^{0,p}(M)$, hence $\delta = L_a$. ■

Schouten bracket

COROLLARY 1: Let $a \in \Lambda^{0,p} \otimes T^{1,0}(M)$, and $b \in \Lambda^{0,q} \otimes T^{1,0}(M)$, **Then there exists $c \in \Lambda^{0,p+q} \otimes T^{1,0}(M)$, such that**

$$\{\operatorname{Lie}_a, \operatorname{Lie}_b\} := \operatorname{Lie}_a \operatorname{Lie}_b - (-1)^{pq} \operatorname{Lie}_b \operatorname{Lie}_a = \operatorname{Lie}_c .$$

Proof: Indeed, the supercommutator of derivations is a derivation, and $\{\operatorname{Lie}_a, \operatorname{Lie}_b\}$ vanishes on antiholomorphic forms, hence it is equal to Lie_c by the previous claim. ■

DEFINITION: The operation taking a, b to $c \in \Lambda^{0,p+q} \otimes T^{1,0}(M)$ is called **the Schouten bracket of $a \in \Lambda^{0,p} \otimes T^{1,0}(M)$, and $b \in \Lambda^{0,q} \otimes T^{1,0}(M)$.**

REMARK: Since $\{\cdot,\cdot\}$ is $\overline{\mathcal{O}}_M$-linear, the Schouten bracket satisfies the Leibnitz identity: $\bar\partial(\{\alpha,\beta\}) = \{\bar\partial\alpha, \beta\} + \{\alpha, \bar\partial\beta\}$.

REMARK: This allows one to extend the Schouten bracket to the $\bar\partial$-cohomology of the complex $(\Lambda^{0,*}(M) \otimes T^{1,0}M, \bar\partial)$, which coincide with the cohomology of the sheaf of holomorphic vector fields: $\{\cdot,\cdot\}: H^p(TM) \times H^q(TM) \longrightarrow H^{p+q}(TM)$.

Tian-Todorov lemma

DEFINITION: Assume that M is a complex n-manifold with trivial canonical bundle K_M, and Φ a non-degenerate section of K_M. We call a pair (M,Φ) **a Calabi-Yau manifold**. Substitution of a vector field into Φ gives an isomorphism $TM \cong \Omega^{n-1}(M)$. Similarly, one obtains an isomorphism

$$\Lambda^{0,q}M \otimes \Lambda^p TM \longrightarrow \Lambda^{0,q}M \otimes \Lambda^{n-p,0}M = \Lambda^{n-p,q}M. \qquad (*)$$

Yukawa product $\bullet: \ \Lambda^{p,q}M \otimes \Lambda^{p_1,q_1}M \longrightarrow \Lambda^{p+p_1-n,q+q_1}M$ is obtained from the usual product

$$\Lambda^{0,q}M \otimes \Lambda^p TM \times \Lambda^{0,q_1}M \otimes \Lambda^{p_1}TM \longrightarrow \Lambda^{0,q+q_1}M \otimes \Lambda^{p+p_1}TM$$

using the isomorphism (*).

TIAN-TODOROV LEMMA: Let (M,Φ) be a Calabi-Yau manifold, and

$$\{\cdot,\cdot\}: \ \Lambda^{0,p}(M) \otimes T^{1,0}M \times \Lambda^{0,q}(M) \otimes T^{1,0}M \longrightarrow \Lambda^{0,p+q}(M) \otimes T^{1,0}M.$$

its Schouten bracket. Using the isomorphism (*), we can interpret Schouten bracket as a map

$$\{\cdot,\cdot\}: \ \Lambda^{n-1,p}(M) \times \Lambda^{n-1,q}(M) \longrightarrow \Lambda^{n-1,p+q}(M).$$

Then, for any $\alpha \in \Lambda^{n-1,p}(M)$, $\beta \in \Lambda^{n-1,p_1}(M)$, one has

$$\{\alpha,\beta\} = \partial(\alpha \bullet \beta) - (\partial\alpha) \bullet \beta - (-1)^{n-1+p} \alpha \bullet (\partial\beta),$$

where $\bullet$ denotes the Yukawa product.

Maurer-Cartan equation and deformations

CLAIM: Let (M,I) be an almost complex manifold, and B an abstract vector bundle over $\mathbb{C}$ isomorphic to $\Lambda^{0,1}(M)$. Consider a differential operator $\overline{\partial}$: $C^\infty M \longrightarrow B = \Lambda^{0,1}(M)$ satisfying the Leibnitz rule. Its symbol is a linear map $u: \Lambda^1(M,\mathbb{C}) \longrightarrow B$. Then $B = \frac{\Lambda^1(M,\mathbb{C})}{\ker u} = \Lambda^{0,1}(M)$. Extend $\overline{\partial}: C^\infty M \longrightarrow B$ to the corresponding exterior algebra using the Leibnitz rule:

$$C^\infty M \xrightarrow{\overline{\partial}} B \xrightarrow{\overline{\partial}} \Lambda^2 B \xrightarrow{\overline{\partial}} \Lambda^3 B \xrightarrow{\overline{\partial}} \ldots$$

Then integrability of I is equivalent to $\overline{\partial}^2 = 0$.

Proof: This is essentially the Newlander-Nirenberg theorem. ■

REMARK: Almost complex deformations of I are given by the sections $\gamma \in T^{1,0}M \otimes \Lambda^{0,1}(M)$, with the integrability relation $(\overline{\partial}+\gamma)^2 = 0$ rewritten as **the Maurer-Cartan equation** $\overline{\partial}(\gamma) = -\{\gamma,\gamma\}$. Here $\overline{\partial}(\gamma)$ is identified with the anticommutator $\{\overline{\partial},\gamma\}$, and $\{\gamma,\gamma\}$ is anticommutator of γ with itself, where γ is considered as a $\Lambda^{0,1}(M)$-valued derivation. **This identifies $\{\gamma,\gamma\}$ with the Schouten bracket.**

Holomorphic symplectic Hodge star operator

Define **the holomorphic symplectic $\star$-map**

$$\star: \Lambda^{p,0}(M) \longrightarrow \Lambda^{2n-p,0}(M)$$

via

$$(\alpha, \beta)_\Omega = \frac{\alpha \wedge \star\beta}{\Omega^n}.$$

This is the usual Hodge star operator on $(1,0)$-variables, with the holomorphic volume form used instead of the usual volume form. We extend $\star$-map to $\Lambda^{p,q}(M)$ by $\star(\alpha\wedge\gamma) = \star(\alpha)\wedge\gamma$ for any $(0,p)$-form γ.

LEMMA: Let M be a holomorphic symplectic manifold. Consider the operators $L_\Omega(\alpha) := \Omega\wedge\alpha$, H_Ω acting as multiplication by $n-p$ on $\Lambda^{p,q}(M)$, and $\Lambda_\Omega := \star\Lambda\star$. **Then $L_\Omega, H_\Omega, \Lambda_\Omega$ satisfy the $\mathfrak{sl}(2)$ relations, similar to the Lefschetz triple: $[H_\Omega, L_\Omega] = 2L_\Omega, \quad [H_\Omega, \Lambda_\Omega] = -2\Lambda_\Omega, [L_\Omega, \Lambda_\Omega] = H_\Omega$.** ■

Tian-Todorov lemma for holomorphically symplectic manifolds

Let now Ω be a holomorphically symplectic form on a complex manifold M, $\dim_{\mathbb{C}} M = 2n$. Then $TM \cong \Omega^1 M$. Define **the holomorphic symplectic Schouten bracket** as the bracket

$$\Lambda^{1,p}(M) \times \Lambda^{1,q}(M) \longrightarrow \Lambda^{1,p+q}(M).$$

obtained from the usual Schouten bracket and this identification.

LEMMA: (Tian-Todorov for holomorphically symplectic manifolds) Let (M,Ω) be a holomorphically symplectic manifold, and

$$\{\cdot,\cdot\}_\Omega : \Lambda^{1,p}(M) \times \Lambda^{1,q}(M) \longrightarrow \Lambda^{1,p+q}(M).$$

the holomorphic symplectic Schouten bracket. **Then for any $a, b \in \Lambda^{1,*}(M)$, one has**

$$\{a,b\}_\Omega = \delta(a \wedge b) - (\delta a) \wedge b - (-1)^{\tilde a} a \wedge \delta(b),$$

where $\tilde a$ is parity of a, and $\delta := [\Lambda_\Omega, \partial]$.

Proof: Later today. ■

Graded Jacobi identity for $\{\eta_i, \eta_j\}_\Omega$

COROLLARY: When a, b are $(1,1)$-forms, **one has** $\{a,b\}_\Omega = \partial\Lambda_\Omega(a\wedge b) - \Lambda_\Omega(\partial(a)\wedge b) - (-1)^{\tilde{a}}\Lambda_\Omega(a\wedge\partial(b))$

Proof: These are the only terms which survive, because $\Lambda_\Omega\big|_{\Lambda^{1,*}(M)} = 0$.
■

From this corollary **it follows that $\{\eta_i, \eta_j\}_\Omega$ satisfies the graded Jacobi relation.** Indeed, this quantity is equal to the Schouten bracket, **which is a super-commutator in the Lie algebra of derivations.**

This proves the remaining claim from Lecture 9 (modulo the holomorphic symplectic Tian-Todorov lemma, which is not proven yet).

Tian-Todorov lemma

We use the usual TT-lemma:

LEMMA: (Tian-Todorov lemma)
Let (M,Φ) be a Calabi-Yau manifold, and

$$\{\cdot,\cdot\}: \Lambda^{0,p}(M)\otimes T^{1,0}M \times \Lambda^{0,q}(M)\otimes T^{1,0}M \longrightarrow \Lambda^{0,p+q}(M)\otimes T^{1,0}M.$$

its Schouten bracket. Using the standard isomorphism $\Lambda^{0,p}(M)\otimes T^{1,0}M = \Lambda^{n-1,p}(M)$, we can interpret the Schouten bracket as a map

$$\{\cdot,\cdot\}: \Lambda^{n-1,p}(M)\times \Lambda^{n-1,q}(M) \longrightarrow \Lambda^{n-1,p+q}(M).$$

Then, for any $\alpha\in\Lambda^{n-1,p}(M)$, $\beta\in\Lambda^{n-1,p_1}(M)$, one has

$$\{\alpha,\beta\} = \partial(\alpha\bullet\beta) - (\partial\alpha)\bullet\beta - (-1)^{n-1+p}\alpha\bullet(\partial\beta), \qquad (*)$$

where $\bullet$ denotes the Yukawa product.

Proof: Since the Schouten bracket is a derivation with respect to the Yukawa product, it suffices to prove (*) on the generators of the Yukawa algebra. The same argument which proves the Moser lemma implies that locally there exist coordinates $z_1, ..., z_n$ such that the holomorphic volume form Φ is $dz_1\wedge...\wedge dz_n$. Then $\{\alpha,\beta\}$ and $\partial(\alpha\bullet\beta) - (\partial\alpha)\bullet\beta - (-1)^{n-1+p}\alpha\bullet(\partial\beta)$ can be computed explicitly when α and β are monomials times a function. ■

Proof of holomorphic symplectic Tian-Todorov lemma

LEMMA: (Tian-Todorov for holomorphically symplectic manifolds) Let (M,Ω) be a holomorphically symplectic manifold, and $\{\cdot,\cdot\}_\Omega : \Lambda^{1,p}(M) \times \Lambda^{1,q}(M) \longrightarrow \Lambda^{1,p+q}(M)$ the holomorphic symplectic Schouten bracket. **Then for any $a, b \in \Lambda^{1,*}(M)$, one has**

$$\{a,b\}_\Omega = \delta(a \wedge b) - (\delta a) \wedge b - (-1)^{\tilde{a}} a \wedge \delta(b), \quad (**)$$

where $\tilde{a}$ is parity of a, and $\delta := [\Lambda_\Omega, \partial]$.

Proof. Step 1: Acting by $\star$ on $\Lambda^{*,*}(M)$, we obtain the Yukawa multiplication from the usual multiplication: $\alpha \bullet \beta = \pm \star (\star\alpha \wedge \star\beta)$. Moreover, the Schouten bracket interpreted as in Tian-Todorov lemma gives a map

$$\{\cdot,\cdot\} : \Lambda^{n-1,p}(M) \times \Lambda^{n-1,q}(M) \longrightarrow \Lambda^{n-1,p+q}(M).$$

after twisting by $\star$ becomes the bracket $\{\cdot,\cdot\}_\Omega$ defined above. Then (**) becomes

$$\{a,b\}_\Omega = \star\partial\star(a \wedge b) - (\star\partial\star a) \wedge b - (-1)^{\tilde{a}} a \wedge \star\partial\star(b).$$

Therefore, **(**) would be implied if we prove the following holomorphic symplectic analogue of the Kähler relations:**

$$\star\partial\star = [\Lambda_\Omega, \partial]. \quad (***)$$

Proof of holomorphic symplectic Tian-Todorov lemma (2)

LEMMA: (Tian-Todorov for holomorphically symplectic manifolds) Let (M,Ω) be a holomorphically symplectic manifold, and $\{\cdot,\cdot\}_\Omega: \Lambda^{1,p}(M)\times\Lambda^{1,q}(M)\longrightarrow\Lambda^{1,p+q}(M)$ the holomorphic symplectic Schouten bracket. **Then for any $a,b\in\Lambda^{1,*}(M)$, one has**

$$\{a,b\}_\Omega = \delta(a\wedge b) - (\delta a)\wedge b - (-1)^{\tilde a} a\wedge\delta(b), \quad (**)$$

where $\tilde a$ is parity of a, and $\delta := [\Lambda_\Omega, \partial]$.

Step 1: We reduced this statement to

$$\star\partial\star = [\Lambda_\Omega,\partial]. \quad (***)$$

Step 2: Denote by δ the operator $[\Lambda_\Omega,\partial]$. The formula (***) is local, hence it would suffice to prove it in a coordinate patch. Using the Moser lemma, we choose holomorphic Darboux coordinates such that $\Omega=\sum_{i=1}^n dz_{2i-1}\wedge dz_{2i}$. Let α be a coordinate monomial and f a function. Then $\delta(f\alpha)=\sum_i \frac{\partial f}{\partial z_i} i_{\Omega^{-1}(dz_i)}(\alpha)$, where $\Omega^{-1}(dz_i)$ the vector field dual to dz_i via Ω. Similarly, $\star\partial\star(f\alpha)=\sum_i\frac{\partial f}{\partial z_i}\star e_{dz_i}\star\alpha$, where $e_{dz_i}(\alpha)=dz_i\wedge\alpha$. Therefore, the commutator relation (***) follows from $\star e_{dz_i}\star = i_{\Omega^{-1}(dz_i)}$. This is clear, because $\star e_{dz_i}\star$ is the convolution with a vector field Ω-dual to dz_i. ■