K3 surfaces, assignment 1: Connections, holonomy, Calabi-Yau manifolds

Exercise 1.1. A connection ∇ is called **flat** if its curvature vanishes. Let (B, ∇) be a flat vector bundle over a simply connected manifold. A section b is called **parallel** $\nabla(b) = 0$. Prove that B is generated by parallel sections.

Exercise 1.2. Let (B_1, ∇_1) and (B_2, ∇_2) be vector bundles with connection over $M \in x$, and $\Theta_i \in \Lambda^2 M \otimes \operatorname{End}(B_i)$ their curvature forms. Denote by $\Theta \in \Lambda^2 M \otimes \operatorname{End}(B_1 \otimes B_2)$ the curvature of the connection

$$\nabla(b_1 \otimes b_2) = \nabla_1(b_1) \otimes b_2 + b_1 \otimes \nabla_2(b_2).$$

induced by ∇_1, ∇_2 on $B_1 \otimes B_2$. Prove that for any $u, v \in T_x M$, one has

$$\Theta(u,v)(b_1 \otimes b_2) = \Theta_1(u,v)(b_1) \otimes b_2 + b_1 \otimes \Theta_1(u,v)(b_2).$$

Exercise 1.3. Let (B, ∇) be a bundle over $M \ni x$, and $\operatorname{Hol} \subset GL(T_x M)$ its holonomy group. Assume that a 2-form $\eta \in (T_x T \otimes T_x M)^*$ is Hol-invariant. Prove that it can be extended to a parallel 2-form on B.

Exercise 1.4. Construct a compact 2-dimensional Riemannian manifold with holonomy of the Levi-Civita connection finite.

Exercise 1.5. Let M be a Riemannian manifold equipped with the Levi-Civita connection, and η a parallel differential form on M. Prove that $d\eta = 0$.

Definition 1.1. A $\bar\partial$-closed $(k,0)$-form on a complex manifold is called **holomorphic**.

Exercise 1.6. Let F be an exact holomorphic p-form on a compact p-dimensional complex manifold. Prove that $F = 0$.

Exercise 1.7. Let M be a compact complex surface (not necessarily Kählerian). Prove that all holomorphic forms on M are closed.

Exercise 1.8. Let M be a compact complex surface equipped with a smooth holomorphic map $M \longrightarrow E$. Assume that E and all fibers of M are elliptic curves. Prove that the canonical bundle of M is trivial.

Exercise 1.9. Let $\mathbb{C}^2/\{\pm 1\}$ be the complex variety obtained as a finite quotient of $\mathbb{C}^2$, and $\widetilde{\mathbb{C}^2/\{\pm 1\}}$ its blow-up in zero. Prove that $\widetilde{\mathbb{C}^2/\{\pm 1\}} \cong T^*\mathbb{C}P^1$.

Exercise 1.10. Let $H = \mathbb{C}^2/\mathbb{Z}^4$ be a complex torus, and $H/\{\pm 1\}$ its quotient by the map $x \mapsto -x$. Denote by M the blow-up of $H/\{\pm 1\}$ in 16 singular points. Prove that M is holomorphically symplectic, that is, admits a closed, non-degenerate (2,0)-form.

Hint. Use the previous exercise.

Remark 1.1. This M is called **the Kummer surface**. It is a K3.