K3 surfaces, assignment 2: cohomology and intersection forms

Exercise 2.1. Compute the cohomology algebra of the manifold

a. $\mathbb{C}P^2 \# \mathbb{C}P^2$

b. $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

c. $(S^2 \times S^3) \# (S^2 \times S^3)$

d. $\mathbb{H}P^2$.

Exercise 2.2. Let $M = \mathbb{C}P^n$. For any $n = 1, 2, ...$, find a diffeomorphism of M which changes the orientation, or prove that it does not exist.

Exercise 2.3. Let G be a finite group acting on a compact manifold M, and $X := \frac{M}{G}$. Prove that $H^*(X, \mathbb{R}) = H^*(M, \mathbb{R})^G$, where $H^*(M, \mathbb{R})^G$ denotes the G-invariant part of the cohomology.

Exercise 2.4. Let τ be an involution of a K3 surface M which has no fixed points, and maps any section Ω of K_M to $-\Omega$. The quotient $X := \frac{M}{\tau}$ is called **an Enriques surface**.

a. Prove that $\chi(X) = 0$, and $b_2(X) = 10$, using the same argument as used to show that $b_2(M) = 20$.

b. Write down the Hodge diamond for X.

Exercise 2.5. Let τ be an involution of a K3 surface M, preserving a non-zero section Ω of K_M.

a. Prove that all fixed points of τ are isolated.

b. Prove that the blow-up of $X := \frac{M}{\tau}$ in all fixed points of τ is again a K3 surface.

c. Prove that the number of fixed points of τ is no bigger than 18.

Exercise 2.6. Define the blow-up of a complex manifold in a point, and prove that the blow-up of $\mathbb{C}P^n$ in a point is diffeomorphic to $\mathbb{C}P^n \# \overline{\mathbb{C}P^n}$.

Exercise 2.7. Let (V, q) be a quadratic lattice $V = \mathbb{Z}^n$ with an indefinite, unimodular quadratic form q. Prove that q can be diagonalized.

Exercise 2.8. Let (V, q) be an unimodular quadratic lattice, and $O(V, q)$ the group of automorphisms of $V = \mathbb{Z}^n$ preserving the scalar product. Prove that $O(V, q)$ is finite when q is sign-definite, and infinite when it is indefinite.