K3 surfaces, assignment 3: Kähler structures on homogeneous spaces

3.1 Almost complex, Hermitian and Kähler structures

Definition 3.1. Let M be a manifold. An endomorphism $I \in \operatorname{End}(TM)$, $I^2 = -\mathsf{Id}_{TM}$ is called **an almost complex structure**, and its $\sqrt{-1}$-eigenbundle is denoted as $T^{1,0}M \subset TM \otimes \mathbb{C}$. An almost complex structure I is called **integrable** if $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$. In this case (M, I) is called **a complex manifold**. A Riemannian metric on an almost complex manifold is called **Hermitian** if it is I-invariant.

Exercise 3.1. Let $U = V \oplus W$ be vector spaces. Prove that their Grassmann algebras are decomposed as follows: $\Lambda^n(U) = \bigoplus_{p+q=n} \Lambda^p V \otimes \Lambda^q W$.[1]

Definition 3.2. Consider the eigenvalue decomposition $\Lambda^1(M,\mathbb{C}) = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M)$ associated with the action of I, with $I\Big|_{\Lambda^{1,0}(M)} = \sqrt{-1}$ and $I\Big|_{\Lambda^{0,1}(M)} = -\sqrt{-1}$. It induces the decomposition on the de Rham algebra

$$\Lambda^k(M,\mathbb{C}) = \bigoplus_{p+q=k} \Lambda^p(\Lambda^{1,0}(M)) \otimes \Lambda^q(\Lambda^{0,1}(M))$$

as shown above. The bundles $\Lambda^p(\Lambda^{1,0}(M))$ and $\Lambda^q(\Lambda^{0,1}(M))$ are denoted $\Lambda^{p,0}(M)$ and $\Lambda^{0,q}(M)$, and the component $\Lambda^p(\Lambda^{1,0}(M)) \otimes \Lambda^q(\Lambda^{0,1}(M))$ is denoted $\Lambda^{p,q}(M)$. The decomposition $\Lambda^k(M,\mathbb{C}) = \bigoplus_{p+q=k} \Lambda^{p,q}(M)$ is called **the Hodge decomposition**, the sections of $\Lambda^{p,q}(M)$ are called (p,q)-forms.

Exercise 3.2. Let (M, I) be an almost complex manifold, and h an I-invariant Riemannian form.

a. Prove that $\omega(x, y) = h(Ix, y)$ is a (1,1)-form.

b. (!) Prove that any Hermitian form h is obtained from a (1,1)-form ω such that $\omega(x, Ix) > 0$ for all non-zero tangent vectors $x \in T_m M$.

Exercise 3.3. Prove that any almost complex manifold admits a Hermitian metric.

Exercise 3.4 (*). Let M be a manifold admitting a non-degenerate 2-form. Prove that M admits an almost complex structure.

[1] This decomposition is not multiplicative, because different factors of the tensor product commute, and the odd forms don't commute.

Definition 3.3. Let (M, I, h) be an almost complex Hermitian manifold The form $\omega(x, y) = h(Ix, y)$ is called **the fundamental form** of M. The triple (M, I, ω) is called **a Kähler triple** if I is integrable and ω is closed. In this case M is called **the Kähler manifold**, h **the Kähler metric** and ω **the Kähler form.**

Remark 3.1. Recall that **symplectic manifold** is a manifold equipped with a non-degenerate, closed 2-form. Clearly, the Kähler form is closed and non-degenerate.

Exercise 3.5. Find a complex, compact manifold not admitting a Kähler metric.

Exercise 3.6 (**). Find a complex, compact manifold not admitting a Kähler metric, but admitting a symplectic structure.

3.2 Symmetric spaces

Definition 3.4. Homogeneous space is a manifold with transitive action of a Lie group (often assumed connected).

Exercise 3.7. Let M be a connected manifold with transitive action of a Lie group G, and H be a stabilizer of a point $x \in M$ (in this case, H is called **the isotropy group** of x).

a. Prove that M is identified with the space of orbits G/H.

b. (!) Let $x, y \in M$, and H_x, H_y be the corresponding isotropy groups. Prove that H_x and H_y are conjugate by some element of G.

Exercise 3.8. Let $M = G/H$ be a homogeneous space with compact H. Assume that M is connected and all non-unit $g \in G$ act non-trivially.

a. (!) Prove that M admits a G-invariant Riemannian structure.

b. (*) Prove that the natural map from the isotropy group of x to $GL(T_xM)$ is injective.

Definition 3.5. A tensor on a manifold M is a section of the **tensor bundle** $TM^{\otimes p} \otimes T^*M^{\otimes q}$. Whenever G acts on M by diffeomorphisms, it acts on the space of tensors, because tensors are functorial.

Exercise 3.9 (!). Let $M = G/H$ be a homogeneous space, and H_x the isotropy group of $x \in M$. Construct a bijective correspondence between G-invariant tensors on M and H_x-invariant vectors in $T_xM^{\otimes p} \otimes T^*_xM^{\otimes q}$.

Definition 3.6. A homogeneous space $M = G/H$ is called **a symmetric space** if M admits a G-invariant Riemannian metric and H_x contains an involution ι which acts as $-\mathsf{Id}$ on T_xM.

Definition 3.7. $SO(n)$ denotes the **special orthogonal** group (the group of all orthogonal matrices preserving the orientation). $U(n)$ is **unitary group** (the group of all complex-linear matrices preserving a Hermitian form). $SU(n)$ is intersection of $U(n)$ and $SL(n, \mathbb{C})$.

Exercise 3.10. Consider the spaces S^{2n}, $\mathbb{C}P^n$, $\mathbb{H}P^n$ equipped with the natural action of $SO(2n+1)$, $U(n+1)$ and $Sp(n+1) := GL(n+1, \mathbb{H}) \cap SO(4n+4)$. Prove that they are symmetric spaces.

Exercise 3.11. Consider the oriented Grassmannian $\mathfrak{Gr}_{\mathbb{R}}(p,q) := \frac{SO(p+q)}{SO(p)\times SO(q)}$,

a. (!) Prove that it is a symmetric space when p or q is even.

b. (*) Prove $\mathfrak{Gr}_{\mathbb{R}}(p,q)$ is isometric to a symmetric space for all p, q.

Definition 3.8. An **odd tensor** on a symmetric space is a tensor $\Psi \in TM^{\otimes p} \otimes T^*M^{\otimes q}$ for $p+q$ odd.

Exercise 3.12. Let $M = G/H$ be a symmetric space, and Ψ a G-invariant odd tensor. Prove that $\Psi = 0$.

3.3 Kähler structures on symmetric spaces

Exercise 3.13 (!). Let $M = G/H$ be a symmetric space, and I a G-invariant almost complex structure. Prove that I is integrable.

Exercise 3.14 (!). $M = G/H$ be a symmetric space, I a G-invariant almost complex structure, and h a G-invariant Hermitian form. Prove that (M, I, h) is Kähler.

Exercise 3.15. Construct a structure of symmetric space and a G-invariant complex structure on the following spaces.

a. $\mathbb{C}P^n$ (also prove that it is Kähler).

b. (!) $\mathfrak{Gr}_{\mathbb{R}}(2,n) := \frac{SO(n+2)}{SO(n)\times SO(2)}$

Exercise 3.16. Consider the standard action of $U(n)$ on $\mathbb{C}^n$. Prove that $\mathbb{C}^n$, considered as a real $U(n)$-representation, is irreducible.

Exercise 3.17. Let V be a real irreducible representation of a group G, and h_1, h_2 two G-invariant bilinear symmetric positive definite forms on V. Prove that h_1 is proportional to h_2.

Exercise 3.18 (!). Let $M = G/H$ be a homogeneous space such that the isotropy group H_x acts on the (real) projectivization $\mathbb{P}T_xM$ transitively. Prove that the G-invariant Riemannian metric on M is unique up to a constant multiplier.

Hint. Use the previous exercise

Exercise 3.19 (*). Consider the oriented Grassmannian $\mathcal{G}r_{\mathbb{R}}(p,q) := \frac{SO(p+q)}{SO(p)\times SO(q)}$. Prove that $\mathcal{G}r_{\mathbb{R}}(p,q)$ admits a $SO(p+q)$-invariant metric. Prove that this metric is unique up to a constant multiplier, when $p > 2$ or $q > 2$.

Exercise 3.20. Construct a $U(n+1)$-invariant Hermitian metric on $\mathbb{C}P^n$ (it is called **Fubini-Study metric**).

a. Prove that this metric is unique up to a constant.

b. Prove that it is Kähler.

Definition 3.9. $U(p,q)$ is the group of all complex-linear matrices preserving a pseudo-Hermitian metric h of signature (p,q), with $h(x_1, ..., x_{p+q}) = \sum_{i=1}^{p} |x_i|^2 - \sum_{i=q+1}^{p+q} |x_j|^2$.

Exercise 3.21. a. (!) Construct a $U(1,n)$-invariant metric and complex structure on $M := \frac{U(1,n)}{U(1)\times U(n)}$.

b. (!) Prove that it is Kähler.

c. (*) Prove that M is biholomorphic to an open ball in $\mathbb{C}^n$.

d. (**) Prove that all complex automorphisms of an open ball are isometries with respect to this metric.

Remark 3.2. This metric on an open ball is called **Bergman metric**, or **complex hyperbolic metric**.

Exercise 3.22 (*). Construct an $SO(n+2)$-invariant Kähler structure on the oriented Grassmannian $\mathcal{G}r_{\mathbb{R}}(2,n) := \frac{SO(n+2)}{SO(n)\times SO(2)}$.