# K3 surfaces, assignment 4: differential operators and connections

## 4.1 Differential operators (after Grothendieck)

**Definition 4.1.** Let $R$ be a commutative ring over a field $k$. Given $a \in R$, consider the map $L_a: R \longrightarrow R$ mapping $x$ to $ax$. Define $\operatorname{Diff}^k(R) \subset \operatorname{Hom}_k(R,R)$ inductively as follows. The $\operatorname{Diff}^0(R)$ is the space of all $R$-linear maps from $R$ to $R$, that is, the space of all $L_a$, $a \in R$. The space $\operatorname{Diff}^k(R)$, $k > 0$ is

$$\operatorname{Diff}^k(R) := \{D \in \operatorname{Hom}_k(R,R) \ \mid \ [L_a, D] \in \operatorname{Diff}^{k-1}(R) \quad \forall a \in R.\}$$

The union of all $\operatorname{Diff}^i(R)$ is called **the space of differential operators on** $R$. Differential operators on the ring $C^\infty M$ is called **differential operators on** $M$**,** denoted $\operatorname{Diff}^*(M)$.

**Exercise 4.1.** Let $D^i \in \operatorname{Diff}^i(R)$, $D^j \in \operatorname{Diff}^j(R)$ be differential operators. Prove that the composition $D^i D^j$ lies in $\operatorname{Diff}^{i+j}(R)$.

**Hint.** Use induction and the identity $[v, D^i D^j] = [v, D^i]D^j + D^i[v, D^j]$

**Exercise 4.2.** Let $D^i \in \operatorname{Diff}^i(R)$, $D^j \in \operatorname{Diff}^j(R)$ be differential operators. Prove that the commutator $[D^i, D^j]$ lies in $\operatorname{Diff}^{i+j-1}(R)$.

**Hint.** Use induction and the Jacobi identity

$$[v, [D^i, D^j]] = [[v, D^i], D^j] + [D^i, [v, D^j]].$$

**Definition 4.2.** Let $R$ be a $k$-algebra, and $D: R \longrightarrow A$ a $k$-linear map from $R$ to an $R$-module. It is called **a $k$-derivation**, or just **derivation** if it satisfies the Leibniz rule: $D(xy) = yD(x) + xD(y)$.

**Exercise 4.3.** a. Prove that $D(k) = 0$ for any $k$-derivation on a $k$-algebra (we assume $\operatorname{char} k = 0$).

b. (!) Let $R$ be a finite extension of a field $k$ of characteristic 0. Prove that the space $\operatorname{Der}_k(R,R)$ of derivations vanishes.

**Exercise 4.4 (**).** Let $R$ be the ring of continuous functions on a manifold $M$. Prove that $\operatorname{Der}_{\mathbb{R}}(R,R) = 0$, or find a counterexample.

**Exercise 4.5 (*).** Let $x_1, ..., x_n$ be coordinates on $\mathbb{R}^n$. Prove that any derivation on $C^\infty\mathbb{R}^n$ is written as coordinates as $D(f) = \sum_{i=1}^n f_i \frac{d}{dx_i}$, where $f_i \in C^\infty M$.

**Hint.** Use the Hadamard lemma and an inclusion $D(I^k) \subset I^{k-1}$ (Exercise 4.8).

**Exercise 4.6 (!).** Let $D \in \operatorname{Diff}^1(R)$ be a differential operator of first order. Prove that $D - D(1)$ is a derivation of $R$. Prove that $\operatorname{Diff}^1(R)/\operatorname{Diff}^0(R)$ is isomorphic to the space of derivations of $R$.

**Exercise 4.7.** Let $R = k[t]$ be an algebra of polynomials over a field $k$ of characteristic 0, and $D \in \operatorname{Diff}^k(R)$.

a. Prove that $D$ is uniquely determined by its restriction on polynomials of degree $\leqslant k$.

b. (*) Prove that $\operatorname{Diff}^k(R)$ is a free $k[t]$-module, generated by $\tau_0, \tau_1, ...\tau_k$, where $\tau_i$ maps all $1, t, t^2, t^3, ...t^k$ except $t^i$ to 0, and $t^i$ to 1.

c. (**) Prove that $\operatorname{Diff}^*(\mathbb{R}[t_1, ..., t_n])$ is an algebra freely generated by generators $t_1, ..., t_n$ and $\frac{d}{dt_1}, ..., \frac{d}{dt_n}$ and relations $[t_i, \frac{d}{dt_j}] = \delta_{ij}$.

**Exercise 4.8.** Let $I \subset R$ be an ideal, and $D \in \operatorname{Diff}^k(R)$. Prove that $D(I^{k+1}) \subset I$.

**Hint.** Use induction in $k$ and identity $[D, L_a L_b] = [D, L_a]L_b + L_a[D, L_b]$.

## 4.2 The ring of symbols of differential operators

**Definition 4.3.** Let $R$ be an associative algebra. **(Increasing) filtration** on $R$ is a collection of subspaces $R_0 \subset R_1 \subset R_2 \subset ...$ such that $R_i R_j \subset R_{i+j}$. The natural product map $(R_k/R_{k-1}) \otimes (R_l/R_{l-1}) \longrightarrow R_{k+l}/R_{k+l-1}$ defines an associative product structure on the space $\bigoplus_{i=0}^\infty R_i/R_{i-1}$. The algebra $\bigoplus_{i=0}^\infty R_i/R_{i-1}$ is called **the associated graded algebra** of this filtration.

**Exercise 4.9.** Consider the algebra $\operatorname{Diff}^*(R)$ with its filtration by $\operatorname{Diff}^i(M)$. Prove that its associated graded algebra is commutative.

**Hint.** Use Exercise 4.2.

**Definition 4.4.** This ring is called **the ring of symbols of differential operators**. For any $D \in \operatorname{Diff}^k(R)$, its class in $\frac{\operatorname{Diff}^k(R)}{\operatorname{Diff}^{k-1}(R)}$ is called **the symbol of** $D$.

**Exercise 4.10.** Consider sections of $TM$ as differential operators of the first order.

a. Prove that $TM = \operatorname{Diff}^1 M / \operatorname{Diff}^0 M$.

b. Prove that the multiplication in the ring of symbols defines a surjective, $C^\infty M$-linear map $\mathsf{Sym}^k TM \longrightarrow \operatorname{Diff}^k M / \operatorname{Diff}^{k-1} M$.

c. (!) Prove that this map is an isomorphism.

## 4.3 Connections

**Definition 4.5.** Let $B$ be a vector bundle on a smooth manifold $M$, and

$$\nabla : B \longrightarrow B \otimes \Lambda^1 M$$

a differential operator which satisfies

$$\nabla(fb) = b \otimes df + f\nabla b,$$

for any $f \in C^\infty(M)$ and any $b \in B$. Then $\nabla$ is called **a connection** on $B$. Given a vector field $X$, consider an operator $\nabla_X : B \longrightarrow B$ obtained by the convolution of $\nabla(b)$ with $X$. This operator is called **the covariant derivative** along $X$.

**Exercise 4.11.** Prove that the covariant derivative $\nabla_X$ satisfies the Leibniz rule $\nabla_X(fb) = \langle df, X\rangle b + f\nabla_X b$. Here $\langle df, X\rangle$ denotes the derivative of $f$ along $X$; elsewhere it is denoted by $\operatorname{Lie}_X f$ or $df \lrcorner X$.

**Exercise 4.12.** Let $B$ be a vector bundle on $M$. Suppose that for any vector field $X \in TM$ we are provided with a covariant derivative operator $\nabla_X : B \longrightarrow B$ satisfying the Leibniz rule and $C^\infty M$-linear on $X$. Prove that $\nabla_X$ is obtained from a connection.

**Exercise 4.13.** Prove that the symbol $\operatorname{Symb}(\nabla) \in TM \otimes \operatorname{Hom}(B, \Lambda^1 M \otimes B)$ of a connection is given by an identity map $\mathsf{Id}_{TM} \otimes \mathsf{Id}_B : TM \otimes T^*M \otimes \operatorname{Hom}(B,B)$.

**Exercise 4.14 (!).** Let $D$ be a first order differential operator on a bundle $B$ with symbol $\operatorname{Symb}(D) \in TM \otimes \operatorname{Hom}(B, \Lambda^1 M \otimes B)$ given by an identity map $\mathsf{Id}_{TM} \otimes \mathsf{Id}_B : TM \otimes T^*M \otimes \operatorname{Hom}(B,B)$. Prove that it is a connection.

**Exercise 4.15 (!).** Let $\nabla$ be a connection on $B$, $B^*$ the dual bundle. Prove that there exists a unique operator $\nabla^* : B^* \longrightarrow B^* \otimes \Lambda^1 M$ such that

$$d\langle b, b'\rangle = \langle \nabla b, b'\rangle + \langle b, \nabla^* b'\rangle$$

for any $b \in B, b' \in B^*$. Prove that $\nabla^*$ is a connection on $B^*$.

**Exercise 4.16.** Let $B_1, ..., B_n$ be vector bundles with connections, denoted by $\nabla$ (people often use the same letter $\nabla$ to denote different connections if they are defined on different bundles). Consider the following differential operator

$$\nabla : B_1 \otimes ... \otimes B_n \longrightarrow B_1 \otimes ... \otimes B_n \otimes \Lambda^1 M,$$

$\nabla(b_1 \otimes b_2 \otimes ... \otimes b_n) = \nabla(b_1) \otimes b_2 \otimes ... \otimes b_n + b_1 \otimes \nabla(b_2) \otimes ... \otimes b_n + ... + b_1 \otimes b_2 \otimes ... \otimes \nabla(b_n)$. Prove that $\nabla$ defines a connection on the vector bundle $B_1 \otimes ... \otimes B_n$.

**Remark 4.1.** Previous two exercises show that a connection on a bundle $B$ defines a connection on any tensor power $B^{\otimes n} \otimes (B^*)^{\otimes m}$. This connection is almost always denoted by the same letter.

**Exercise 4.17 (!).** Let $B$ be a vector bundle over a manifold admitting partition of unity. Prove that $B$ admits a connection.

## 4.4 Holonomy

**Definition 4.6.** Let $(B, \nabla)$ be a bundle with connection. A tensor $\Psi \in B^{\otimes i} \otimes (B^*)^{\otimes j}$ is called **parallel** if $\nabla(\Psi) = 0$. In this case we also say that $\Psi$ **is preserved** by $\nabla$.

**Exercise 4.18.** Let $g$ be a tensor on a bundle $B$ over $M$. Construct a connection $\nabla$ such that $\nabla(g) = 0$ if

a. (!) $g$ is a non-degenerate bilinear symmetric form on $B$.

b. (*) $g$ is a non-degenerate bilinear antisymmetric form on $B$.

c. (*) $g$ is a bilinear symmetric form of constant rank on $B$.

**Exercise 4.19.** Let $B$ be a trivial vector bundle with connection over $\mathbb{R}$. Prove that for each $x \in \mathbb{R}$ and each vector $b_x \in B\big|_x$ there exists a unique section $b \in B$ such that $\nabla b = 0$, $b\big|_x = b_x$.

**Definition 4.7.** Let $\gamma : [0,1] \longrightarrow M$ be a smooth path in $M$ connecting $x$ and $y$, and $(B, \nabla)$ a vector bundle with connection. Restricting $(B, \nabla)$ to $\gamma[0,1]$, we obtain a bundle with connection on an interval. Solve an equation $\nabla(b) = 0$ for $b \in B\Big|_{\gamma([0,1])}$ and initial condition $b\big|_x = b_x$. This process is called **parallel transport** along the path via the connection. The vector $b_y := b\Big|_y$ is called **vector obtained by parallel transport of $b_x$ along $\gamma$.** **Holonomy group** of $\gamma$ is the group of endomorphisms of the fiber $B_x$ obtained from parallel transports along all paths starting and ending in $x \in M$

**Exercise 4.20.** Let $(B, \nabla)$ be a vector bundle over a connected manifold $M$, and $x, y \in M$. Construct an isomorphism of the corresponding holonomy groups $\operatorname{Hol}_x(\nabla) \longrightarrow \operatorname{Hol}_y(\nabla)$.

**Exercise 4.21.** Find a bundle with connection over $S^1$ which has non-trivial holonomy.

## 4.5 Iterated connection

**Definition 4.8.** Let $M$ be a smooth manifold. A connection on $TM$ or on $\Lambda^1 M$ is called **connection on** $M$. This connection defines a connection on all tensor powers of $TM$ and $\Lambda^1 M$. A tensor product of several copies of $TM$ and $\Lambda^1 M$ is called **a tensor bundle** on $M$, and its section **a tensor**. Similarly, a section of a tensor product of several copies of $B$ and $B^*$ is called **a tensor over a bundle** $B$.

**Definition 4.9.** Let $B$ be a vector bundle with connection $\nabla_0$ over a manifold $M$, and $\nabla$ a connection on $\Lambda^1 M$. Define a connection

$$\nabla_i : \ B \otimes \underbrace{\Lambda^1 M \otimes ...}_{i \text{ times}} \longrightarrow B \otimes \underbrace{\Lambda^1 M \otimes ...}_{i+1 \text{ times}} \tag{4.1}$$

using the Leibniz formula

$$\nabla_i(b \otimes \xi_1 \otimes ... \otimes \xi_i) = \nabla_{i-1}(b \otimes \xi_1 \otimes ... \otimes \xi_{i-1}) \otimes \xi_i + b \otimes \xi_1 \otimes ... \otimes \xi_{i-1} \otimes \nabla \xi_i.$$

Denote by

$$\nabla^i : \ B \longrightarrow B \otimes \underbrace{\Lambda^1 M \otimes ...}_{i \text{ times}}$$

the composition $\nabla_0 \circ \nabla_1 \circ ... \circ \nabla_i$. This operator is called **an $i$-th power of the connection** $\nabla$.

**Exercise 4.22.** a. Prove that the symbol of $\nabla^2$, considered as an element of

$$\mathsf{Sym}^2 TM \otimes \operatorname{Hom}(B, B \otimes \Lambda^1 M \otimes \Lambda^1 M)$$

is symmetric under the permutation of the tensor multipliers $\Lambda^1 M \otimes \Lambda^1 M$.

b. (*) Let $S$ be the symbol of $\nabla^i$,

$$S \in \mathsf{Sym}^i TM \otimes \operatorname{Hom}\left(B, B \otimes \underbrace{\Lambda^1 M \otimes ...}_{i \text{ times}}\right)$$

Prove that $S$ is symmetric under the permutations of the tensor multipliers $\Lambda^1 M \otimes \Lambda^1 M \otimes ... \otimes \Lambda^1 M$.

c. (*) Prove that $S$ is given by $\mathsf{Id} \in \operatorname{End}(\mathsf{Sym}^i \Lambda^1 M \otimes B)$, where the bundle $\operatorname{End}(\mathsf{Sym}^i \Lambda^1 M \otimes B)$ is identified with $\mathsf{Sym}^i TM \otimes \operatorname{Hom}(B, B \otimes \mathsf{Sym}^i \Lambda^1 M)$ using an isomorphism $V \otimes \operatorname{Hom}(B, B \otimes V^*) = \operatorname{Hom}(B \otimes V^*, B \otimes V^*)$, where $V = \mathsf{Sym}^i TM$.

**Exercise 4.23.** Let $D \in \operatorname{Diff}^s(B, C)$ be a differential operator on vector bundles $B$, $C$. Prove that there exists a $C^\infty$-linear map

$$\Psi : \ B \otimes \bigoplus_{i=0}^{s} (\Lambda^1 M)^{\otimes i} \longrightarrow C$$

such that $D(b) = \Psi\left(\bigoplus_{i=0}^s \nabla^i b\right)$.