K3 surfaces, assignment 5: Ehresmann connections and foliations

5.1 Ehresmann connections

Definition 5.1. Let $\pi : M \longrightarrow Z$ be a smooth fibration, with $T_\pi M$ **the bundle of vertical tangent vectors** (vectors tangent to the fibers of π). An **Ehresmann connection** on π is a sub-bundle $T_{\mathsf{hor}} M \subset TM$ such that $TM = T_{\mathsf{hor}} M \oplus T_\pi M$. The **parallel transport** along the path $\gamma : [0,a] \longrightarrow Z$ associated with the Ehresmann connection is a diffeomorphism

$$V_t : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(t))$$

smoothly depending on $t \in [0,a]$ and satisfying $\frac{dV_t}{dt} \in T_{\mathsf{hor}} M$.

Exercise 5.1 (*). Let $\pi : M \longrightarrow X$ be a continuous map of manifolds. Prove that π is proper (preimages of compacts are compact) if and only if it is closed (images of closed sets are closed) and all its fibers are compact.

Exercise 5.2. Let $\pi : M \longrightarrow X$ be a smooth submersion of manifolds. Prove that when π is proper, all fibers of π are diffeomorphic. Find an example when π is non-proper and its fibers are not diffeomorphic.

Exercise 5.3. Let M, Z be smooth manifolds, and $\pi : M \longrightarrow Z$ a proper submersion.

a. Prove that for any smooth path $\gamma : [0,a] \longrightarrow Z$, the parallel transport map exists, and is defined uniquely.

b. Prove that π is **locally trivial**, that is, for a sufficiently small $U \subset Z$ there exists a decomposition $\pi^{-1}(U) = F \times U$, and this identification carries π to the projection $F \times U \longrightarrow U$.

Remark 5.1. This statement is called **the Ehresmann theorem**: every proper smooth submersion defines a locally trivial fibration. Further on, the term “proper smooth submersion” is used as a synonym of “locally trivial smooth fibration”.

Definition 5.2. Let $\pi_1 : M_1 \longrightarrow Z$, $\pi_2 : M_2 \longrightarrow Z$ be any maps. The **fibered product** $M_1 \times_Z M_2$ is

$$M_1 \times_Z M_2 := \{(u,v) \in M_1 \times M_2 \mid \pi_1(u) = \pi_2(v)\}.$$

Exercise 5.4. Let $\pi_1 : M_1 \longrightarrow Z$, $\pi_2 : M_2 \longrightarrow Z$ be smooth submersions with fibers F_1, F_2. Prove that the fibered product $M_1 \times_Z M_2 \longrightarrow Z$ a smooth submersion with fiber $F_1 \times F_2$.

Exercise 5.5. Let $\pi_1 : M_1 \longrightarrow Z$, $\pi_2 : M_2 \longrightarrow Z$ be smooth submersions with fibers F_1, F_2, and $TM_i = T_{\sf hor} M_i \oplus T_\pi M_i$ the Ehresmann connections. Let $\pi : M_1 \times_Z M_2 \longrightarrow Z$ be the standard projection.

a. Consider the exact sequence

$$0 \longrightarrow T_\pi(M_1 \times_Z M_2) \longrightarrow T(M_1 \times_Z M_2) \longrightarrow \pi^* TZ \longrightarrow 0. \qquad (5.1)$$

Let $\Pi_1 : M_1 \times_Z M_2 \longrightarrow M_1$, $\Pi_2 : M_1 \times_Z M_2 \longrightarrow M_2$ be the projection maps. Consider a sub-bundle

$$B := \{v \in T_{m_1,m_2}(M_1 \times_Z M_2) \mid D\Pi_1(v) \in T_{\sf hor} M_1, \text{ and } D\Pi_2(v) \in T_{\sf hor} M_2\}.$$

Prove that B is isomorphic to $\pi^* TZ$ and defines a splitting of the exact sequence (5.1), $T(M_1 \times_Z M_2) = B \oplus T_\pi(M_1 \times_Z M_2)$, that is, an Ehresmann connection on the fibration $M_1 \times_Z M_2$.

b. Conversely, consider an Ehresmann connection $T(M_1 \times_Z M_2) = T_\pi(M_1 \times_Z M_2) \oplus T_{\sf hor}(M_1 \times_Z M_2)$. Prove that image of the differential $D\Pi_i(T_{\sf hor}(M_1 \times_Z M_2)) \subset TM_i$ satisfies $D\Pi_i(T_{\sf hor}(M_1 \times_Z M_2)) \oplus T_{\pi_i} M_i = TM_i$, that is, defines an Ehresmann connection on the fibration $\pi_i M_i \longrightarrow Z$.

c. Construct a bijective correspondence between the Ehresmann connections on the fibration $\pi : M_1 \times_Z M_2 \longrightarrow Z$ and the pairs of Ehresmann connections on $\pi_1 : M_1 \longrightarrow Z$, $\pi_2 : M_2 \longrightarrow Z$.

Definition 5.3. Let B be a vector bundle on M and $\pi : \operatorname{Tot} B \longrightarrow M$ its total space. An Ehresmann connection $T \operatorname{Tot} B = T_{\sf hor} \operatorname{Tot} B \oplus T_\pi \operatorname{Tot} B$ on π is called **homogeneous** if the decomposition $T \operatorname{Tot} B = T_{\sf hor} \operatorname{Tot} B \oplus T_\pi \operatorname{Tot} B$ is preserved by the differential of homothety map $v \longrightarrow \lambda v$, that is, if the differential of homothety map takes horizontal tangent vectors to horizontal tangent vectors. Consider the addition map $\operatorname{Tot} B \times_M \operatorname{Tot} B \longrightarrow \operatorname{Tot} B$, and let $\tilde\nabla$ denote the Ehresmann connection on $\operatorname{Tot} B \times_M \operatorname{Tot} B \longrightarrow \operatorname{Tot} B$ induced by the Ehresmann connection on $\operatorname{Tot} B$ as in Exercise 5.5. We say that an Ehresmann connection $T \operatorname{Tot} B = T_{\sf hor} \operatorname{Tot} B \oplus T_\pi \operatorname{Tot} B$ is **additive** if the differential of the addition map $\operatorname{Tot} B \times_M \operatorname{Tot} B \longrightarrow \operatorname{Tot} B$ takes horizontal vectors to horizontal vectors. We say that an Ehresmann connection is **linear** if it is additive and homogeneous.

Exercise 5.6. Let B be a vector bundle on M and $\pi : \operatorname{Tot} B \longrightarrow M$ its total space. Prove that an Ehresmann connection on $\operatorname{Tot} B$ is linear of and only if the parallel transport map $V_t : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(t))$ taking a fiber of B to another fiber of B is linear with respect to the structure of the vector space on these fibers.

Definition 5.4. Let $M \longrightarrow Z$ be a locally trivial fibration equipped with an Ehresmann connection. The projection $\pi : M \longrightarrow Z$ defines an isomorphism $T_{\sf hor} M\big|_x \xrightarrow{\sim} T_{\pi(x)} Z$. Therefore, for each vector field $X \in TZ$, there exists a unique vector field $X_{\sf hor} \in T_{\sf hor} M$ such that $d\pi(X_{\sf hor}) = X$, called **the horizontal lift** of X.

Exercise 5.7. Prove that an Ehresmann connection on $\operatorname{Tot} B$ is linear if and only if the flow V_t of any horizontal lift $X_{\mathsf{hor}} \in T_{\mathsf{hor}} \operatorname{Tot} B$ is compatible with the structure of vector space on the fibers of π.

Exercise 5.8. Let $T \operatorname{Tot} B = T_{\mathsf{hor}} \operatorname{Tot} B \oplus T_\pi \operatorname{Tot} B$ be an Ehresmann connection, and $\operatorname{Tot} B^* \times_M \operatorname{Tot} B \xrightarrow{\kappa} \operatorname{Tot} C^\infty M = \mathbb{C} \times M$ the natural pairing. Prove that there exists a unique Ehresmann connection on $\operatorname{Tot} B^*$ such that the map κ takes horizontal vectors to horizontal vectors.

Definition 5.5. We call the Ehresmann connection on $\operatorname{Tot} B^*$ defined in Exercise 5.8 **the dual Ehresmann connection**.

Definition 5.6. Let B be a vector bundle, and f a function on $\operatorname{Tot} B$ which is linear on all fibers of π. Such a function is called **fiberwise linear**.

Exercise 5.9. Prove that an Ehresmann connection on $\operatorname{Tot} B$ is linear if and only if the Lie derivative of a fiberwise linear function along a horizontal vector field is again fiberwise linear.

Hint. The diffeomorphism flow associated with a horizontal lift induces a linear map on fibers if and only if the Ehresmann connection is linear.

Exercise 5.10. Prove that every section of B defines a fiberwise linear function on $\operatorname{Tot} B^*$, and, conversely, every fiberwise linear function on $\operatorname{Tot} B^*$ is associated with a smooth section of B.

Exercise 5.11. Let $\operatorname{Tot} B$ be a total space of a vector bundle equipped with an linear Ehresmann connection, and ∇ the dual Ehresmann connection on $\operatorname{Tot} B^*$ (Exercise 5.8). For a vector field X on M, denote by $\tilde{X} \in T \operatorname{Tot} B^*$ its horizontal lift. Consider a section b of B as a fiberwise linear function on $\operatorname{Tot} B^*$. Let $\nabla_X b := \operatorname{Lie}_{\tilde{X}} b$ be the fiberwise linear function on B^* obtained as a horizontal lift of X. Using Exercises 5.9, 5.10, we interpret $\nabla_X b$ as a section of B. Prove that $b \longrightarrow \nabla_X b$ defines a connection operator on the vector bundle B.

Exercise 5.12. Let B be a smooth vector bundle on a manifold M. Construct a natural bijective correspondence between the linear Ehresmann connections on $\operatorname{Tot} B$ and the connection operators on B.

Hint. Use the previous exercise.

5.2 Frobenius theorem

Definition 5.7. **A distribution** on a manifold is a sub-bundle $B \subset TM$.

Exercise 5.13. Let $\Pi: TM \longrightarrow TM/B$ be the projection, and $x, y \in B$ some vector fields. Prove that the map $x, y \longrightarrow \Pi([x,y])$ is linear in x and y.

Definition 5.8. The map $[B,B] \longrightarrow TM/B$, putting x, y to $\Pi([x,y])$, is called **Frobenius bracket** (or **Frobenius form**); it is a skew-symmetric $C^\infty(M)$-linear form on B with values in TM/B.

Definition 5.9. A distribution is called **integrable**, or **Frobenius integrable**, or **holonomic**, or **involutive**, if its Frobenius form vanishes.

Remark 5.2. Let $B \subset TM$ be a sub-bundle. **Frobenius theorem** claims that B is involutive if and only if each point $x \in M$ has a neighbourhood $U \ni x$ and a smooth submersion $U \xrightarrow{\pi} V$ such that B is its vertical tangent space: $B = T_\pi M$. We shall prove it later in this assignment.

Definition 5.10. The fibers of π are called **leaves**, or **integral submanifolds** of the distribution B. Globally on M, **a leaf of** B is a maximal connected manifold $Z \hookrightarrow M$ which is immersed to M and tangent to B at each point. The set of leaves of an integrable distribution is called **a foliation**. The leaves are manifolds which are immersed in M. [1]

Remark 5.3. Let $U \subset M$ be an open subset. A leaf of the foliation $\mathcal{F}$ in $U \subset M$ is a connected component of $F \cap U$, where F is a leaf of $\mathcal{F}$ in M. However, the intersection $F \cap U$ can possibly have many connected components. This means that the set of leaves of $\mathcal{F}$ on U is different from the set of leaves of $\mathcal{F}$ on M.

Remark 5.4. If B is the tangent bundle to a foliation, then the condition $[B,B] \subset B$ is clear, because it is true leaf-wise. To prove the Frobenius theorem we need only to prove the existence of the foliation tangent to B, for any involutive $B \subset TM$.

Remark 5.5. To prove the Frobenius theorem for $B \subset TM$, it suffices to show that each point is contained in an integral submanifold. In this case, the smooth submersion $U \xrightarrow{\pi} V$ is the projection to the leaf space of B.

Exercise 5.14. Let G be a Lie group acting on a manifold M. Assume that the vector fields from the Lie algebra of G generate a sub-bundle $B \subset TM$. Prove that Frobenius theorem holds for B.

[1] The leaves are immersed, but not necessarily closed. Quite often it occurs that some (or all) leaves of a foliation are dense in M.

Hint. Prove that the leaves of the corresponding foliation are the orbits of the G-action.

Exercise 5.15. Let u be a non-vanishing vector field on a manifold M. Prove that locally around every point of M there exists a coordinate system with coordinate functions $x_1, ..., x_n$ such that $u = \frac{\partial}{\partial x_1}$.

Exercise 5.16. Let u, v be commuting vector fields on a manifold M, and e^{tu}, e^{tv} be corresponding diffeomorphism flows. Prove that e^{tu}, e^{tv} commute.

Hint. Use the previus exercise, express v in coordinates $x_1, ..., x_n$, and prove that the coefficients of v depend only on $x_2, ..., x_n$.

Remark 5.6. Let $B \subset TM$ be a distribution such that $[B, B] \subset B$. We are going to show that locally B has a basis $\xi_1, ..., \xi_k$ of commuting vector fields. By the previous exercise, the diffeomorphisms $e^{t\xi_i}$ generate a finite-dimensional commutative Lie group acting locally on M, and by Exercise 5.14, Frobenius theorem holds true for B.

Exercise 5.17. Let $\sigma: M \longrightarrow M_1$ be a smooth submersion,

$$D\sigma: T_xM \longrightarrow T_{\sigma(x)}M_1$$

its differential, and $v \in TM$ a vector field which satisfies

$$D\sigma(v)\Big|_x = D\sigma(v)\Big|_y \tag{5.2}$$

for any $x, y \in \sigma^{-1}(z)$ and any $z \in M_1$. Clearly, in this case $D(\sigma(v))$ is a well-defined vector field on M_1. Let $u, v \in TM$ be vector fields which satisfy (5.2). Prove that the commutator $[u, v]$ also satisfies (5.2), and, moreover, $D\sigma([u, v]) = [D\sigma(u), D\sigma(v)]$.

Exercise 5.18. Let $B \subset TM$ be any distribution, and $m \in M$ any point.

a. Prove that there exists a neighbourhood $U \ni m$ and a smooth submersion $\sigma: U \longrightarrow W$ such that $\ker D\sigma \oplus B = TM$.

b. For any vector field $X \in TW$, prove that there exists a unique vector field $\tilde{X} \in B$ satisfying (5.2). We call $\tilde{X}$ **a B-lift of** X

c. Let $X, Y \in TW$ be commuting vector fields, and $\tilde{X}, \tilde{Y}$ their B-lifts. Prove that $[\tilde{X}, \tilde{Y}] \in \ker D\sigma$

Hint. Use the previous exercise.

Exercise 5.19. Let $B \subset TM$ be a distribution which satisfies $[B, B] \subset B$, and $\sigma: M \longrightarrow M_1$ a smooth submersion such that $TM = \ker D\sigma \oplus B$. Consider commuting vector fields $X, Y \in TM_1$, and let $\tilde{X}, \tilde{Y}$ be their B-lifts. Prove that $[\tilde{X}, \tilde{Y}] = 0$.

Hint. Use the previous exercise.

Exercise 5.20. Let $B \subset TM$ be a distribution which satisfies $[B, B] \subset B$. Prove that every point $m \in M$ has a neighbourhood U such that the restriction of B to U has a basis of commuting vector fields.

Hint. Use Exercise 5.18 and Exercise 5.19.

Exercise 5.21. Prove Frobenius theorem: for any distribution $B \subset TM$ which satisfies $[B, B] \subset B$, and any point $m \in M$ has a neighbourhood U which admits a smooth submersion $\pi : U \longrightarrow V$ such that $B = \ker D\pi$.

Hint. Use Exercise 5.20 and Remark 5.6.

Exercise 5.22 (*). Construct a 4-manifold M and a rank 2 distribution $B \subset TM$ such that $[B, B]$ has rank 3 and $[[B, B], B]$ has rank 4.

Exercise 5.23 (*). Let $M = \mathbb{R}$. Find two vector fields X, Y on M such that the successive commutators of X, Y generate an infinite-dimensional Lie algebra.

Exercise 5.24. Construct a 4-dimensional smooth manifold not admitting rank 3 distributions.

Exercise 5.25. Prove that a compact n-dimensional torus admits a rank one foliation with all leaves dense.