K3 surfaces, assignment 6: Local systems

6.1 Sheaves and vector bundles

Definition 6.1. Let M be a topological space. **A sheaf** $\mathcal{F}$ on M is a collection of sets, abelian groups or vector spaces $\mathcal{F}(U)$ defined for each open subset $U \subset M$, with the **restriction maps**, which are linear homomorphisms $\mathcal{F}(U) \stackrel{\phi_{U,U'}}{\longrightarrow} \mathcal{F}(U')$, defined for each $U' \subset U$, and satisfying the following conditions.

(A) Composition of restrictions is again a restriction: for any open subsets $U_3 \subset U_2 \subset U_1$, the corresponding restriction maps

$$\mathcal{F}(U_1) \stackrel{\phi_{U_1,U_2}}{\longrightarrow} \mathcal{F}(U_2) \stackrel{\phi_{U_2,U_3}}{\longrightarrow} \mathcal{F}(U_3)$$

give $\phi_{U_1,U_2} \circ \phi_{U_2,U_3} = \phi_{U_1,U_3}$.[1]

(B) Let $U \subset M$ be an open subset, and $\{U_i\}$ a cover of U. For any $f \in \mathcal{F}(U)$ such that all restrictions of f to U_i vanish, one has $f = 0$.

(C) Let $U \subset M$ be an open subset, and $\{U_i\}$ a cover of U. Consider a collection $f_i \in \mathcal{F}(U_i)$ of sections, defined for each U_i, and satisfying

$$f_i\Big|_{U_i\cap U_j} = f_j\Big|_{U_i\cap U_j}$$

for each U_i, U_j. Then there exists $f \in \mathcal{F}(U)$ such that the restriction of f to U_i is f_i.

The space $\mathcal{F}(U)$ is called **the space of sections of the sheaf $\mathcal{F}$ on U**. The restriction maps are often denoted $f \longrightarrow f\Big|_U$.

Exercise 6.1. Let $\mathcal{C}$ be the category of open subsets of a topological space M, with morphisms given by embeddings of a open subset $U \subset M$ to an open subset containign U. Prove that a presheaf is a functor from $\mathcal{C}$ to the category of vector spaces.

[1] If (A) is satisfied, $\mathcal{F}$ is called **a presheaf**. **Morphism of presheaves** $\mathcal{F} \longrightarrow \mathcal{G}$ is a collection of maps $\mathcal{F}(U) \longrightarrow \mathcal{G}(U)$, defined for all open subsets $U \subset M$, which commute with the restriction maps.

Exercise 6.2. Let M be a topological space equipped with a presheaf of abelian groups $\mathcal{F}$. Prove that the conditions (B) an (C) are equivalent to exactness of the following sequence

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \xrightarrow{\Psi} \prod_{i\neq j} \mathcal{F}(U_i \cap U_j)$$

for any open $U \subset M$, and any cover $\{U_i\}$ of U. Here the map Ψ takes $\prod_i f_i$, $f_i \in \mathcal{F}(U_i)$ to $\prod_{i\neq j} \left[\psi_j(f_i) - \psi_i(f_j)\right]$, where $\psi_j : \mathcal{F}(U_i) \longrightarrow \mathcal{F}(U_i \cap U_j)$ denotes the restriction map.

Definition 6.2. A sheaf homomorphism $\psi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ is a collection of homomorphisms

$$\psi_U : \mathcal{F}_1(U) \longrightarrow \mathcal{F}_2(U),$$

defined for each $U \subset M$, and commuting with the restriction maps. **A sheaf isomorphism** is a homomorphism $\Psi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$, for which there exists an homomorphism $\Phi : \mathcal{F}_2 \longrightarrow \mathcal{F}_1$, such thate $\Phi \circ \Psi = \mathsf{Id}$ and $\Psi \circ \Phi = \mathsf{Id}$.

Exercise 6.3. Let $\psi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ be a sheaf homomorphism.

a. Show that $U \longrightarrow \ker \psi_U$ and $U \longrightarrow \operatorname{coker} \psi_U$ are presheaves.

b. Prove that $U \longrightarrow \ker \psi_U$ is a sheaf (it is called **the kernel** of a homomorphism ψ).

c. (*) Prove that $U \longrightarrow \operatorname{coker} \psi_U$ is not always a sheaf (find a counterexample).

Definition 6.3. A subsheaf $\mathcal{F}' \subset \mathcal{F}$ is a sheaf associating to each $U \subset M$ a subspace $\mathcal{F}'(U) \subset \mathcal{F}(U)$.

Exercise 6.4. Find a non-zero sheaf $\mathcal{F}$ on M such that $\mathcal{F}(M) = 0$.

Exercise 6.5. a. Given a covering $\{U_i\}$ of U and a presheaf $\mathcal{F}$ on U, denote by $\mathcal{F}(\{U_i\})$ the kernel of the map $\prod_i \mathcal{F}(U_i) \xrightarrow{\Psi} \prod_{i\neq j} \mathcal{F}(U_i \cap U_j)$ (Exercise 6.2). Prove that for any refinement $\{V_j\}$ of the covering $\{U_i\}$, there is a functorial map $\mathcal{F}(\{U_i\}) \longrightarrow \mathcal{F}(\{V_j\})$.

b. Let $\mathcal{F}$ be a presheaf on M. Denote by $\mathcal{F}_1(U)$ the direct limit of $\mathcal{F}(\{U_i\})$ taken over all sequences of successive refinements of the coverings of U. Prove that this limit is well defined. Prove that $U \mapsto \mathcal{F}_1(U)$ defines a presheaf on M.

c. Prove that $\mathcal{F}_1(U)$ satisfies the axiom C of the definition of sheaves (Definition 6.1).

d. Denote by $\mathcal{F}_2(U)$ the quotient of $\mathcal{F}_1(U)$ by the union K_U of the kernels of the map $\mathcal{F}_1(U) \longrightarrow \prod_i \mathcal{F}_1(U_i)$ for all open coverings $\{U_i\}$ of U. Prove that the restriction map takes K_U to K_V for any open set $V \subset U$, Deduce that the restriction maps $\frac{\mathcal{F}_1(U)}{K_U} \longrightarrow \frac{\mathcal{F}_1(V)}{K_V}$ define a structure of presheaf on $U \mapsto \mathcal{F}_2(U)$

e. Prove that $\mathcal{F}_2$ is a sheaf.

Definition 6.4. The sheaf $\mathcal{F}_2$ obtained from the presheaf $\mathcal{F}$ as above is called **the sheafification** of $\mathcal{F}$.

Exercise 6.6. Let $\mathcal{F}$ be a sheaf, $\mathcal{F}_2$ its sheafification.

a. Prove that the natural maps $\mathcal{F}(U) \longrightarrow \mathcal{F}_1(U) \longrightarrow \mathcal{F}_2(U)$ define a morphism of the presheaf $\mathcal{F}$ to its sheafification.

b. Prove that for any presheaf morphism $\phi: \mathcal{F} \longrightarrow \mathcal{G}$ to a sheaf $\mathcal{G}$, the morphism ϕ factorizes through the sheafification $\mathcal{F}_2$.

Remark 6.1. Let $A \xrightarrow{\phi} B$ be a ring homomorphism, and V a B-module. Then V is equipped with a natural A-module structure: $av := \phi(a)v$.

Definition 6.5. **A sheaf of rings** on a manifold M is a sheaf $\mathcal{F}$ with all the spaces $\mathcal{F}(U)$ equipped with a ring structure, and all restriction maps ring homomorphisms.

Definition 6.6. Let $\mathcal{F}$ be a sheaf of rings on a topological space M, and $\mathcal{B}$ another sheaf. It is called **a sheaf of $\mathcal{F}$-modules** if for all $U \subset M$ the space of sections $\mathcal{B}(U)$ is equipped with a structure of $\mathcal{F}(U)$-module, and for all $U' \subset U$, the restriction map $\mathcal{B}(U) \xrightarrow{\phi_{U,U'}} \mathcal{B}(U')$ is a homomorphism of $\mathcal{F}(U)$-modules (use Remark 6.1 to obtain a structure of $\mathcal{F}(U)$-module on $\mathcal{B}(U')$).

Exercise 6.7. Let $\mathcal{F}_1$ be a sheaf of rings and $\mathcal{F}$ its subsheaf, closed under multiplication (that is, $\mathcal{F}$ is a ring subsheaf). Prove that $\mathcal{F}_1$ is a sheaf of modules over $\mathcal{F}$.

Definition 6.7. A free sheaf of modules $\mathcal{F}^n$ over a ring sheaf $\mathcal{F}$ maps an open set U to the space $\mathcal{F}(U)^n$. A sheaf of $\mathcal{F}$-modules is **non-free** if it is not isomorphic to a free sheaf.

Exercise 6.8 (!). Find a subsheaf of modules in $C^\infty M$ which is non-free in the sense of this definition.

Definition 6.8. Locally free sheaf of modules over a sheaf of rings $\mathcal{F}$ is a sheaf of modules $\mathcal{B}$ satisfying the following condition. For each $x \in M$ there exists a neighbourhood $U \ni x$ such that the restriction $\mathcal{B}\Big|_U$ is free.

Definition 6.9. A smooth vector bundle on a smooth manifold M is a locally free sheaf of $C^\infty M$-modules.

Exercise 6.9. Let B be a vector bundle on M. Prove that B can be always equipped with a smooth Euclidean metric.

Exercise 6.10. Let M be a compact smooth manifold.

a. Prove that M admits an embedding to $\mathbb{R}^n$.

b. Prove that its tangent bundle can be obtained as a direct summand of a trivial vector bundle.

Hint. Prove that TM is embedded to $T\mathbb{R}^n\Big|_M$ and use the previous exercise.

Exercise 6.11. Let B be a vector bundle on M, $B_1 := B \oplus C^\infty M$, and $M_1 := \mathbb{P}B_1$ its projectivization. Consider an embedding $\tau : \ M \longrightarrow M_1$ taking $x \in M$ to a point $(0:1)$ of the fiber $\mathbb{P}\left(B \oplus C^\infty M\Big|_x\right)$.

a. Prove that $\tau^*(TM_1) \cong B \oplus TM$.

b. Prove that any vector bundle on a compact smooth manifold can be obtained as a direct summand of a trivial vector bundle.

Hint. Apply Whitney embedding theorem to M_1 and use the previous exercise.

Exercise 6.12 (**). Let M be a smooth compact manifold. Recall that a module over a ring R is called **projective** if it is a direct summand of a free module $R^I = \bigoplus_{i\in I} R$. Let $\Gamma(C^\infty M)$ be the ring of smooth functions on M. Serre-Swann theorem claims that the category of vector bundles on M is equivalent to the category of projective $\Gamma(C^\infty M)$-modules. Define the morphisms in both categories so that this statement makes sense, and prove it.

6.2 Germs and pullbacks

Definition 6.10. Let $Z \subset M$ be a subset of a topological space (not necessarily open or closed), and $\mathcal{F}$ a sheaf on M. Two sections f_1, f_2 of $\mathcal{F}$ defined in two open neighbourhoods U_1, U_2 of Z **have equivalent germs in** Z if for a sufficiently small neighbourhood U of Z, with $U \subset U_1 \cap U_2$, one has $f_1\Big|_U = f_2\Big|_U$. The set of such equivalence classes is called **the space of germs of sections of $\mathcal{F}$ in Z;** it is equipped with a natural group structure.

Exercise 6.13. Let $\mathcal{B}$ be a sheaf on M such that the space of germs of $\mathcal{B}$ in all points of M is equal 0. Prove that $\mathcal{B} = 0$.

Exercise 6.14. Find a sheaf $\mathcal{F}$ on M with the space of germs in all points of M non-zero, and $\mathcal{F}(M)$ zero.

Definition 6.11. A sheaf $\mathcal{F}$ on M is called **soft** if for any closed subset $X \subset M$, the natural map from the space of global sections $\mathcal{F}(M)$ to the space of germs $\mathcal{F}_g(X)$ is surjective.

Exercise 6.15. Show that the sheaf of real analytic functions on $\mathbb{R}^n$ is not soft.

Exercise 6.16. Show that a constant sheaf on a manifold of positive dimension is not soft.

Exercise 6.17. Find a topological space M and a functions $\mathcal{F}$ on it such that the restriction map from $\mathcal{F}(M)$ to the space of germs of $\mathcal{F}$ in a point is always surjective, but the sheaf $\mathcal{F}$ is not soft.

Exercise 6.18. Let $N, N' \subset M$ be two closed subsets of a metric space, $N \cap N' = \emptyset$. Prove that there exist non-intersecting neighbourhoods $U \supset N$, $U' \supset N'$.

Exercise 6.19. Let M be a manifold admitting a partition of unity, $N \subset M$ a closed subset, and $U \supset N$ its neighbourhood. Prove that M has a locally finite cover $\{U_i\}$, such that all U_i which intersect N have compact closures and satisfy $\bar{U}_i \subset U$.

Hint. Prove that M admits a metric, and use the previous exercise.

Definition 6.12. **Support** of a function f is the set of all points where $f \neq 0$. A function is called **supported in** U if its support is contained in U.

Exercise 6.20. Let $U \subset M$ be an open subset of a manifold, $U' \subset M$ an open subset with compact closure satisfying $\bar{U}' \subset U$, and f a smooth function on U with support in U'. Prove that f can be extended to a smooth function on M.

Exercise 6.21 (*). Let M be a manifold admitting a partition of unity. Prove that the sheaf of smooth functions on M is soft.

Hint. Given a smooth function f on $U \supset N$, find a cover $\{U_i\}$, $i \in I$ as in Exercise 6.19, and let $\{\psi_i\}$ be a subordinate partition of unity. Let $A \subset I$ be the set of indices $\alpha \in I$ such that $U_\alpha \cap N \neq 0$. Prove that the function $f' := \sum_{\alpha\in A} \psi_\alpha f$ is supported in $U' \Subset U$, can be extended smoothly to the whole M, and equal f on N.

Definition 6.13. Let $\mathcal{F}$ be a sheaf on X, and $f: \ Y \longrightarrow X$ a continuous map. Let $\mathcal{F}_g(Z)$ denote the space of germs of $\mathcal{F}$ in $Z \subset X$. Define **the pullback presheaf** $f^*(\mathcal{F})$ as $f^*(\mathcal{F})(U) := \mathcal{F}_g(f(U))$. Since $\mathcal{F}_g(Z)$ is equipped with a natural restriction map $\mathcal{F}_g(Z) \longrightarrow \mathcal{F}_g(Z')$ for any $Z' \subset Z$, compatible with successive embeddings $Z'' \subset Z' \subset Z$, the map $U \longrightarrow f^*(\mathcal{F})(U)$ defines a presheaf. We define **the pullback sheaf** as the sheafification of this presheaf.

Exercise 6.22. Find an example when the pullback presheaf is not a sheaf.

Exercise 6.23. Let $f: \ Y \longrightarrow X$ be a continuous map, and $\mathcal{F}$ a sheaf on X. Prove that the germ space of $f^*\mathcal{F}$ in $y \in Y$ is equal to the germ space of $\mathcal{F}$ in $f(y)$.

Exercise 6.24. Prove that the functor $\mathcal{F} \to f^*(\mathcal{F})$ **is exact**, that is, takes an exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

to an exact sequence

$$0 \longrightarrow f^*\mathcal{F}_1 \longrightarrow f^*\mathcal{F}_2 \longrightarrow f^*\mathcal{F}_3 \longrightarrow 0.$$

6.3 Local systems

Definition 6.14. A constant sheaf on a on a topological space M is a sheaf $\mathcal{F}$ such that for any connected open set, $\mathcal{F}(U) = A$, where A is a fixed vector space, and the corresponding restricion maps are isomorphisms. **A locally constant sheaf** is a sheaf $\mathcal{F}$ such that for each $x \in M$ there exists a neighbourhood $U \ni x$ such that the restriction $\mathcal{F}\Big|_U$ is a constant sheaf. **A local system** is a locally constant sheaf of abelian groups or vector spaces.

Exercise 6.25. Let M be a manifold, and $\mathfrak{O}(U)$ takes U to the set of all orientations on U. Prove that $\mathfrak{O}(M)$ is a locally constant sheaf of sets. Prove that this sheaf is constant when M is orientable and non-constant when M is not orientable.

Exercise 6.26. Let $\mathcal{F}$ be a locally constant sheaf on M, $x \in M$ a point, and $\mathcal{F}_x$ the space of germs of $\mathcal{F}$ in x. Prove that $\mathcal{F}_x = \mathcal{F}(U)$ for any sufficiently small connected neighbourhood containing x.

Exercise 6.27. Conversely, let $\mathcal{F}$ be a sheaf such that the natural map $\mathcal{F}(U) \longrightarrow \mathcal{F}_x$ is an isomorphism for any sufficiently small neighbourhood of x. Prove that $\mathcal{F}$ is locally constant.

Exercise 6.28. Prove that a pullback of a locally constant sheaf is locally constant.

Exercise 6.29. Prove that any locally constant sheaf on an interval $[0,1]$ is constant.

Exercise 6.30. Prove that any locally constant sheaf on an the square $[0,1] \times [0,1]$ is constant.

Definition 6.15. Let $\mathcal{F}$ be a locally constant sheaf on S^1, and $[0,1] \xrightarrow{\tau} S^1$ the map gluing the ends together. Since the sheaf $\tau^*\mathcal{F}$ is locally constant on $[0,1]$, it is constant (Exercise 6.29). This can be used to construct an isomorphism of the germ spaces $\Psi: \tau^*\mathcal{F}_0 \longrightarrow \tau^*\mathcal{F}_1$. Another isomorphism, denoted by Φ, is produced by identifying $\tau^*\mathcal{F}_1$, $\tau^*\mathcal{F}_0$ and $\mathcal{F}_0$. The composition $\Psi \otimes \Phi^{-1}: \mathcal{F}_0 \to \mathcal{F}_0$ is called **the monodromy of the local system on S^1**. It is obtained by taking a section of $\mathcal{F}(\tau(]0,\varepsilon[))$, moving it along S^1 by identifying naturally $\mathcal{F}(]x,y[)$ and $\mathcal{F}(]x+\delta, y+\delta[)$ for $\delta < |x-y|$, and going the full circle.

Exercise 6.31 (!). Let $\mathcal{R}$ be the functor taking a local system $\mathcal{F}$ on S^1 to a representation of $\mathbb{Z}$ on $\mathcal{F}_0$ defined by the monodromy. Prove that this functor defines an equivalence of categories between the local systems on S^1 and representations of $\mathbb{Z}$.

Exercise 6.32. Let $\mathcal{F}$ be a local system on a manifold M, and $\gamma: [0,1] \longrightarrow M$ a smooth embedding.

a. Prove that for a sufficiently small neighbourhood U of $\operatorname{im}\gamma$, the restriction $\mathcal{F}\Big|_U$ is a constant sheaf.

b. Prove that an isomorphism of the germ spaces $\mathcal{F}_{\gamma(0)} \tilde{\to} \mathcal{F}_{\gamma(1)}$ induced by the trivialization of $\mathcal{F}\Big|_U$ is independent from the choice of U.

Exercise 6.33. Let $\gamma, \gamma': [0,1] \longrightarrow M$ be smooth embeddings which satisfy $\gamma(0) = \gamma'(0)$ and $\gamma(1) = \gamma'(1)$. Assume that these paths are homotopic. Applying Exercise 6.32 to γ and γ', we obtain two isomorphisms of germ spaces $\mathcal{F}_{\gamma(0)} \tilde{\to} \mathcal{F}_{\gamma(1)}$. Prove that these isomorphisms are equal.

Exercise 6.34. Let M be a connected, simply connected manifold. Prove that any locally constant sheaf on M is constant.

Hint. Use the previous exercise.

Exercise 6.35 (!). Let $\gamma: S^1 \longrightarrow M$ be a loop taking 0 to x, $\mathcal{F}$ a local system on M, and $\mathcal{F}_x$ its germ space. Denote by $\chi_\gamma \in \operatorname{Aut}(\mathcal{F}_x)$ the monodromy of the pullback $\gamma^*(\mathcal{F})$, considered as a locally constant sheaf on S^1.

a. Prove that the map χ_γ is uniquely determined by the homotopy class of γ.

b. Prove that χ_γ defines a homomorphism from $\pi_1(M,x)$ to $\operatorname{Aut}(\mathcal{F}_x)$, that is, satisfies $\chi_{\gamma\gamma'} = \chi_\gamma \circ \chi_{\gamma'}$.

Definition 6.16. The homomorphism $\pi_1(M,x) \to \operatorname{Aut}(\mathcal{F}_x)$, defined in the previous exercise, is called **the monodromy of the local system $\mathcal{F}$**.

Exercise 6.36 (*). Prove that the monodromy defines an equivalence between the category of local systems and the category of representations of $\pi_1(M,x)$.