K3 surfaces, assignment 7: Riemann-Hilbert correspondence

7.1 Derivations

Remark 7.1. All rings in these handouts are assumed to be commutative and with unit. **Rings over a field** k are rings containing a field k.

Definition 7.1. Let R be a ring over a field k. A k-linear map $D:\ R \longrightarrow R$ is called **a derivation** if it satisfies **the Leibnitz equation** $D(fg) = D(f)g + gD(f)$. The space of derivations is denoted as $\operatorname{Der}_k(R)$.

Exercise 7.1. Let $D \in \operatorname{Der}_k(R)$. Prove that $D\Big|_k = 0$.

Exercise 7.2. Let D_1, D_2 be derivations. Prove that the commutator $[D_1, D_2] := D_1D_2 - D_2D_1$ is also a derivation.

Exercise 7.3 (!). Let $K \supset k$ be a field which contains a field k of characteristic 0, and is finite-dimensional over k (such fields K are called **finite extensions** of k). Prove that $\operatorname{Der}_k(K) = 0$.

Exercise 7.4 (*). Is it true if $\mathsf{char}\, k = p$?

Exercise 7.5. Consider a ring $k[\varepsilon]$, given by a relation $\varepsilon^2 = 0$. Find $\operatorname{Der}_k(k[\varepsilon])$.

Exercise 7.6 (*). Find all rings R over $\mathbb{C}$ such that R is finite-dimensional over $\mathbb{C}$, and $\operatorname{Der}_{\mathbb{C}}(R) = 0$.

Exercise 7.7 (*). Consider the polynomial ring $\mathbb{R}[x_1, ..., x_n]$ as a subring of $C^\infty(\mathbb{R}^n)$. Prove that any derivation of $C^\infty(\mathbb{R}^n)$ is uniquely determined by its restriction to $\mathbb{R}[x_1, ..., x_n]$. Prove that any $C^\infty(\mathbb{R}^n)$-valued derivation on $\mathbb{R}[x_1, ..., x_n]$ defines a derivation of $C^\infty(\mathbb{R}^n)$.

7.2 Curvature of a vector bundle

Exercise 7.8. For any vector field X on M, denote by Lie_X the corresponding derivation on $C^\infty M$.

a. Prove that for any vector fields $X, Y \in TM$, there exists a vector field Z which satisfies $\operatorname{Lie}_Z = [\operatorname{Lie}_X, \operatorname{Lie}_Y]$.

b. Prove that Z is determined uniquely from this relation.

Definition 7.2. The vector field Z constructed as above is called **the commutator** of X and Y, denoted $[X, Y]$.

Exercise 7.9. Let (B, ∇) be a vector bundle with connection on a manifold M, b its section, and $X, Y \in TM$ two vector fields. Prove that $\nabla_X \nabla_Y b - \nabla_Y \nabla_X b - \nabla_{[X,Y]} b$ is linear in X, Y, b.

Definition 7.3. Let (B, ∇) be a vector bundle with connection on a manifold M. **The curvature** of ∇ is an $\operatorname{End}(B)$-valued 2-form which is given by $\Theta(X, Y)(b) = \nabla_X \nabla_Y b - \nabla_Y \nabla_X b - \nabla_{[X,Y]} b$. A bundle is called **flat** if its curvature vanishes.

Definition 7.4. Let B be a vector bundle, and Ψ a section of its tensor power. We say that **connection ∇ preserves Ψ** if $\nabla(\Psi) = 0$. In this case, we also say that the tensor Ψ is **parallel** with respect to the connection.

Remark 7.2. $\nabla(\Psi) = 0$ is equivalent to Ψ being a solution of $\nabla(\Psi) = 0$ on each path γ. This means that the parallel transport preserves Ψ.

Exercise 7.10. Let (B, ∇) be a bundle which is locally generated by parallel sections. Prove that ∇ is flat.

Remark 7.3. One of the main purposes of the present assignment is to prove the converse statement: a flat bundle is locally generated by parallel sections.

7.3 Curvature of an Ehresmann connection

Definition 7.5. A vector field v on $\mathbb{R}^n$ is called **linear** if it satisfies $v\Big|_{x+y} = v\Big|_x + v\Big|_y$ and $v\Big|_{\lambda x} = \lambda v\Big|_x$ for any $x, y \in \mathbb{R}^n$.

Exercise 7.11. Prove that a vector field $v \in T\mathbb{R}^n$ is linear of and only if for any affine function v on $\mathbb{R}^n$, the derivative $\operatorname{Lie}_v f$ is affine.

Exercise 7.12. Prove that a commutator of linear vector fields is linear.

Exercise 7.13. Let $\mathfrak{g}$ be the space of linear vector fields on $V = \mathbb{R}^n$, and W the space of affine functions on $\mathbb{R}^n$.

a. Prove that the derivations along linear vector fields preserve W and act trivially on constants.

b. Use this observation to construct a homomorphism from the Lie algebra $\mathfrak{g}$ to $\operatorname{End}(V^*)$.

Exercise 7.14. Let $\mathfrak{g}$ be the space of linear vector fields on $\mathbb{R}^n$. Construct a bijective correspondence between $\mathfrak{g}$ and $\operatorname{End}(\mathbb{R}^n)$ which is compatible with commutators.

Hint. Use the previous exercise.

Definition 7.6. Let $\pi: \operatorname{Tot} B \longrightarrow M$ be a total space of a vector bundle. A vertical vector field $v \in T_\pi \operatorname{Tot}(B)$ is called **fiberwise linear** if it is linear on all fibers.

Remark 7.4. From Exercise 7.14, we obtain that the fiberwise linear vector fields are in bijective correspondence with sections of $\operatorname{End}(B)$. For any $E \in \operatorname{End}(B)$, the corresponding vector field v associates to $x \in \operatorname{Tot}(B)$ the vector $E(v) \in B\Big|_x \subset T_x \operatorname{Tot}(B)$.

Exercise 7.15. Let $M = \mathbb{R}^n$ and $X \in TM$ a vector field such that for any linear vector field, the commutator $[X, Y]$ is linear. Prove that X is also linear.

Exercise 7.16. Let $\pi: P \longrightarrow M$ be a smooth submersion. **A lift** of a vector field $X \in TM$ is a vector field $\tilde{X}$ on P such that for all $x \in P$ the differential $d\pi$ takes $\tilde{X}\Big|_x \in T_x P$ to $X\Big|_{\pi(x)} \in T_{\pi(x)} M$.

a. Consider a $C^\infty(M)$-linear map $l: \Gamma(TM) \longrightarrow \Gamma(TP)$ which maps each vector field to its lift. Prove that $\operatorname{im} l$ is a sub-bundle in TP.

b. Prove that $TP = T_\pi P \oplus \operatorname{im}(l)$.

c. Prove that for every Ehresmann connection $TP = T_\pi P \oplus T_{\mathsf{hor}} P$, and any vector field $X \in TM$ there exists a unique lift $\tilde{X} \in T_{\mathsf{hor}} P$.

Definition 7.7. Let $TP = T_\pi P \oplus T_{\mathsf{hor}} P$ be an Ehresmann connection on a smooth fibration $\pi: P \longrightarrow M$. **The horizontal lift** of a vector field $X \in TM$ is its lift $\tilde{X} \in T_{\mathsf{hor}} P$. It exists and is unique by Exercise 7.16.

Exercise 7.17. Let $TP = T_\pi P \oplus T_{\mathsf{hor}} P$ be an Ehresmann connection on a smooth fibration $\pi: \ P \longrightarrow M$, $X, Y \in TM$ vector fields, and $\tilde X, \tilde X$ their horizontal lift. Denote by $\widetilde{[X,Y]}$ the horizontal lift of $[X,Y]$. Prove that $[\tilde X, \tilde Y] - \widetilde{[X,Y]}$ is a vertical vector field.

Definition 7.8. In these assumptions, the map taking $X, Y \in TM$ to $[\tilde X, \tilde Y] - \widetilde{[X,Y]} \in T_\pi P$ is called **the curvature of the Ehresmann connection on π.**

Exercise 7.18. Consider a smooth fibration π over a 1-dimensional manifold. Prove that the curvature of any Ehresmann connection in π vanishes.

Exercise 7.19. Let B be a vector bundle on M, $\pi: \ \operatorname{Tot} B \longrightarrow M$ its total space, and $T \operatorname{Tot} B = T_\pi \operatorname{Tot} B \oplus T_{\mathsf{hor}} \operatorname{Tot} B$ an Ehresmann connection on $\operatorname{Tot} B$.

a. Prove that this connection is linear if and only if for any fiberwise linear function $\lambda \in C^\infty(\operatorname{Tot} B)$, and any vector field $X \in TM$, which is horizontally lifted to $\tilde X$, the function $\operatorname{Lie}_{\tilde X} \lambda$ is fiberwise linear.

b. Prove that $\nabla_X(\lambda) := \operatorname{Lie}_{\tilde X} \lambda$ defines a connection ∇^* on the bundle B^* of fiberwise smooth functions on $\operatorname{Tot} B$.

c. In Assignment 5, it was shown that the linear Ehresmann connections on $\pi: \ \operatorname{Tot} B \longrightarrow M$ are in bijective correspondence with connections ∇ on B. Show that the connection ∇^* satisfies

$$\operatorname{Lie}_X \langle \lambda, b \rangle = \langle \nabla^*_X \lambda, b \rangle + \langle \lambda, \nabla_X b \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between B and B^*, λ is a section of B^* and b a section of B.

Remark 7.5. The following exercise shows that the curvature of a connection on a vector bundle B is equal to the curvature of the corresponding Ehresmann connection on $\operatorname{Tot} B$.

Exercise 7.20. Let B be a vector bundle with connection ∇, $\pi: \ \operatorname{Tot} B \longrightarrow M$ its total space, equipped with the linear Ehresmann connection induced by ∇, and λ a fiberwise linear function on $\operatorname{Tot} B$, considered as a section of B^*. Consider commuting vector fields $X, Y \in TM$, and let $\tilde X$, $\tilde Y$ be their horizontal lifts.

a. Prove that $[\nabla_X, \nabla_Y](\lambda) = \operatorname{Lie}_{[\tilde X, \tilde Y]} \lambda$.

b. Prove that the vector field $[\tilde X, \tilde Y]$ is fiberwise linear.

c. Let $\Theta \in \Lambda^2(M) \otimes \operatorname{End}(B)$ be the curvature of B. Identifying $\operatorname{End}(B)$ and fiberwise linear vector fields, we may consider Θ as a map taking $X, Y \in TM$ to a fiberwise linear vector field $\Theta(X,Y) \in T_\pi \operatorname{Tot} B$. Prove that $\Theta(X,Y) = [\tilde X, \tilde Y]$

Hint. Use Exercise 7.19 to relate the connection in B and the Ehresmann connection in $\operatorname{Tot} B$ and apply the relation $[\nabla_X, \nabla_Y](\lambda) = \operatorname{Lie}_{[\tilde X, \tilde Y]} \lambda$ to relate the curvature of ∇ and the curvature of the Ehresmann connection.

7.4 Riemann-Hilbert correspondence

Exercise 7.21. Let B be a vector bundle with connection ∇, and $\pi : \operatorname{Tot} B \longrightarrow M$ its total space, equipped with the linear Ehresmann connection $T \operatorname{Tot} B = T_\pi \operatorname{Tot} B \oplus T_{\mathsf{hor}} \operatorname{Tot} B$ induced by ∇.

a. Prove that a section of π, considered as a submanifold in $\operatorname{Tot}(B)$, is tangent to $T_{\mathsf{hor}} \operatorname{Tot} B$ if and only if it is parallel.

b. Prove that the Frobenius form of the distribution $T_{\mathsf{hor}} \operatorname{Tot} B \subset T \operatorname{Tot} B$ vanishes if and only if the curvature of ∇ vanishes.

c. Suppose that ∇ is flat. Prove that for every point $x \in \operatorname{Tot} B$ there exists a neighbourhood $U \subset M$ of $\pi(x)$ and a parallel section of $B\Big|_U$ passing through x.

Hint. For the last statement, use the Frobenius theorem.

Exercise 7.22. Let (B, ∇) be a flat vector bundle on M

a. Prove that any point $x \in M$ has a neighbourhood U such that the restriction $B\Big|_U$ has a basis $b_1, ..., b_n$ such that $\nabla(b_i) = 0$.

b. Suppose that $b_1, ..., b_n$ and $b'_1, ..., b'_n$ are such bases, satisfying $\nabla(b_i) = \nabla(b'_i) = 0$. Prove that there exists a matrix $(a_{ij}) \in GL(n, \mathbb{R})$ such that $b_j = \sum_{j=1}^n b'_i a_{ij}$.

Exercise 7.23 (!). Let (B, ∇) be a flat vector bundle on M, and $\mathbb{B}$ the sheaf of all parallel sections of B. Prove that the sheaf $\mathbb{B}$ is locally constant.

Exercise 7.24. Let (B, ∇_0) be a trivial line bundle with the standard connection on $M := \mathbb{C}\backslash 0$, and $\nabla := \nabla_0 + \frac{dz}{z}$. Denote by $\mathbb{B}$ the corresponding locally constant sheaf. Prove that $\mathbb{B}$ is non-trivial, and find its monodromy.

Exercise 7.25. Let $\mathbb{B}$ be a locally constant sheaf of vector spaces on a manifold M, considered as a sheaf of modules over a constant sheaf $\mathbb{R}_M$

a. Prove that the sheaf $B := \mathbb{B} \otimes_{\mathbb{R}_M} C^\infty M$ is a locally free sheaf of $C^\infty M$-modules, that is, a vector bundle.

b. Let $U \subset M$ be an open set such that the restriction $\mathbb{B}\Big|_U$ is trivial, and $b_1, ..., b_n$ a basis in $\mathbb{B}(U)$. A general section of $B\Big|_U$ can be written as $\sum_{i=1}^n f_i b_i$, where $f_i \in C^\infty U$. Define the connection $\nabla : B \longrightarrow B \otimes \Lambda^1 U$ in $B\Big|_U$ by $\nabla \left(\sum_{i=1}^n f_i b_i\right) = \sum b_i \otimes df_i$. Prove that this connection is flat and independent from the choice of an open set and the basis $b_1, ..., b_n$.

Exercise 7.26. (Riemann-Hilbert correspondence)
Prove that the functors constructed earlier in this subsection define an equivalence between the category of flat vector bundles[1] and the category of locally constant sheaves of vector spaces.

[1]The objects of this category are pairs (B, ∇), where ∇ is a flat connection on a vector bundle B; morphisms are sheaf morphisms compatible with the connections.